Prop'n

(a) $kY \to Y$ is continuous

(b) If X is a k-space,
$f: X \to Y$ C^0 $\iff$ $f: X \to kY$ C^0.

(c) If X is a k-space, and $\sim$ an equiv relation,
$X/\sim$ is a k-space.

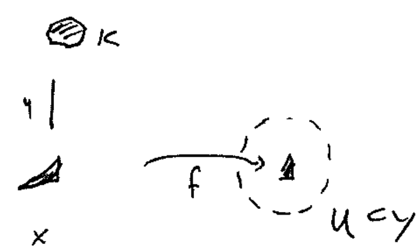
(d) If X is a k-space, and $\{Y_\alpha\}$ a collection of k-spaces,
$$X \longrightarrow k(\prod_\alpha^{usual} Y_\alpha)$$
is continuous iff
$$X \longrightarrow k(\prod^{usual} Y_\alpha) \longrightarrow Y_\alpha$$
is.

(e) Let X be a k-space. $U \subset X$ is open iff
$\forall$ $u: K \to X$, $u^{-1}(U) \subset K$ is open.

Exer Let X be compact, and fix $y \in Y$. (Y arbitrary space.)
Also fix $U \subset X \times^{\text{usual}} Y$ s.t. $X \times \{y\} \subset U$, U open.
Then $\exists$ $V \subset Y$ open, $y \in V$, s.t. $X \times^{\text{usual}} V \subset U$.

Defn Let X, Y be arbitrary spaces.

$\forall$ • K compact Haus, ~~maps~~
• maps $u: K \to X$ cts,
• $U \subset Y$ open,



define
$$W(u, K, U) = \{ f: X \to Y \ C^0 \ \text{s.t.} \ fu(K) \subseteq U \}.$$

Let $\mathrm{Maps}'(X,Y)$ be space w/ coarsest topology s.t. $W(u,K,U)$ is open $\forall$ (u,K,U).

Let
$$\mathrm{Maps}(X,Y) := k\,\mathrm{Maps}'(X,Y).$$

Rmk The topology of Maps' is called the <u>compact-open</u> topology.

Lemma

(a) Fix maps $q : W \to X$, $t : Y \to Z$ C^0,

w/ W, X, Y, Z arbitrary spaces.

The maps

$$t_* : \mathrm{Maps}(X,Y) \longrightarrow \mathrm{Maps}(X,Z)$$
$$f \longmapsto t \circ f$$

$$q^* : \mathrm{Maps}(X,Y) \longrightarrow \mathrm{Maps}(W,Y)$$
$$f \longmapsto f \circ q$$

are continuous.

(b) Let X and Y be k-spaces.

The maps

$$\mathrm{ev} : X \times \mathrm{Maps}(X,Y) \longrightarrow Y$$
$$(x, f) \longmapsto f(x)$$

$$\mathrm{inj} : Y \longrightarrow \mathrm{Maps}(X, X \times Y)$$
$$y \longmapsto (x \mapsto (x,y))$$

are C^0.

(c) Consider the isomorphisms

$$\hom_{Sets}(X \times Y, Z) \rightleftarrows \hom_{Sets}(X, \hom_{Sets}(Y,Z))$$
$$\alpha \longmapsto (x \mapsto \alpha(x,-)) \qquad \cancel{x \in X,\ \beta(x)(y) =}$$
$$(\beta(-))(-) \longleftarrow \beta$$

~~Then $\beta(x)(y)$ is C^0 iff~~

Then $\alpha = (\beta(\cdot))(\cdot)$ is C^0 iff

- $\beta(x) : Y \to Z$ is C^0 $\forall x$
- The function $x \longmapsto \beta(x)$ is C^0.

Pf (a) Given $u: K \to X$, $U \subset Z$,

$$(t_*)^{-1}(W(u,K,U)) = t_*^{-1}\{g: X \to Z \mid gu(K) \subset U\}$$
$$= \{f: X \to Y \mid tfu(K) \subset U\}$$
$$= \{f: X \to Y \mid fu(K) \subset t^{-1}U\}$$
$$= W(u, K, t^{-1}U). \qquad \text{So } t_* \text{ is } C^0.$$

Moreover, $W(u,K,t^{-1}U)$ open in $\mathcal{M}aps'$ $\Rightarrow$ open in $\mathcal{M}aps$.

$\Rightarrow$ $\mathcal{M}aps(X,Y) \to \mathcal{M}aps'(X,Z)$ C^0

$\Rightarrow$ $\mathcal{M}aps(X,Y) \to k\mathcal{M}aps'(X,Z) = \mathcal{M}aps(X,Z)$ C^0.

Likewise, given $u: K \to W$, $U \subset Y$,

$$(q^*)^{-1} W(u,K,U) = (q^*)^{-1}\{g: W \to Y \mid gu(K) \subset U\}$$
$$= \{f: X \to Y \mid fqu(K) \subset U\}$$
$$= W(qu, K, U).$$

(b) inj: NTS $\forall$ $u: K \to X$, $U \subset X \times Y$, $\text{inj}^{-1}(W(u,K,U)) \subset Y$ is open.

i.e., $\forall$ $v: L \to Y$, $v^{-1}\,\text{inj}^{-1}\,W(u,K,U) \subset L$ open. (L cmpct Haus).

Well, u, v $C^0 \Rightarrow$ $u \times v: K \times L \to X \times Y$ C^0

$\Rightarrow$ $(u \times v)^{-1}U \subset K \times L$ open

$\Rightarrow$ $\{\ell \in L \mid K \times \{\ell\} \subset (u \times v)^{-1}U\} \subset L$ is open.

But $K \times \{\ell\} \subset (u \times v)^{-1}U \iff \ell \in v^{-1}\,\text{inj}^{-1}\,W(u,K,U)$.

ev: NTS $\forall$ $u: K \to X \times \mathcal{M}aps(X,Y)$, $U \subset Y$, $u^{-1}\text{ev}^{-1}U \subset K$ open.

Consider
$$\begin{tikzcd} K \arrow[r] \arrow[dr, "u_X"'] \arrow[rr, bend left, "u_M"] & X \times \mathcal{M}aps(X,Y) \arrow[d] \arrow[r] & \mathcal{M}aps(X,Y) \\ & X & \end{tikzcd}$$
. So $p \in u^{-1}\text{ev}^{-1}U \iff (u_M p)(u_X p) \in U$.

Since $u_M(p): X \to Y$ is C^0, $u_M(p) \circ u_X : K \to Y$ is C^0.

Since K Haus, $\exists$ $L \subset K$ closed, $p \in L$ s.t. $L \subset (u_M(p) \circ u_X)^{-1}(U)$ and $\exists$ $p \in V$,

So $u_M(p) \in W(u_X, L, U)$. Since u_M is C^0 by assumption, $u_M^{-1}W(u_X, L, U) \subset K$ is open

By constrxn, $u_M^{-1}(W(u_X, L, U)) \cap L \subset u^{-1}\text{ev}^{-1}U$, and $p \in u_M^{-1}W(u_X,L,U) \cap (V \subset L)$. //

(b) ev:

Need to show

$$\forall \quad u : K \to X \times \mathrm{Maps}(X,Y),$$

$$U \subset Y \text{ open},$$

we have $\quad u^{-1} ev^{-1} U$ open.

(Since $X \times \mathrm{Maps}(X,Y)$ is k-space, this shows $ev^{-1} U$ is open.)

$$\begin{tikzcd} K \arrow[d, "u"] \\ X \times \mathrm{Maps}(X,Y) \arrow[d, "ev"] \\ Y \end{tikzcd}$$

Consider

$$\begin{tikzcd} & K \arrow[dl, "x"'] \arrow[d] \arrow[dr, "f"] & \\ X & X \times \mathrm{Maps}(X,Y) \arrow[l, "\pi_X"] \arrow[r, "\pi_M"'] & \mathrm{Maps}(X,Y) \end{tikzcd}$$

We'll study $x^{-1} f_p^{-1} U \cap f^{-1} W$ for some W.

So $\quad p \in u^{-1} ev^{-1} U \iff f_p(x_p) \in U. \quad (p \in K.)$

Since f_p is C^0, $\quad f_p \circ x : K \to Y$ is C^0.

Since K Haus, $\exists \quad \underset{\text{open}}{V} \subset \underset{\text{compact}}{L} \subset K, \; x \in V$, s.t.

$$L \subset (f_p \circ x)^{-1}(U).$$

So $\quad f_p \in W(x|_L, L, U) \subset \mathrm{Maps}(X,Y)$.

Since $f : K \to \mathrm{Maps}(X,Y)$ is C^0, $\quad f^{-1}(W(x|_L, L, U)) \subset K$ is open.

Hence $\quad V \cap f^{-1}(W(x|_L, L, U)) \subset K$ is open, and contains p.

So $\quad u^{-1} ev^{-1} U$ is open.

Pf (c) Suppose β satisfies

$$\beta(x) : Y \to Z \text{ is } C^0 \quad \forall x \in X$$

$$\beta : X \to \mathrm{Maps}(Y, Z) \text{ is } C^0.$$

Then α is

$$X \times Y \xrightarrow{\beta \times id} \mathrm{Maps}(Y, Z) \times Y \xrightarrow{ev} Z.$$

Since ev is C^0 by (b), and β C^0 by assump,

$$\alpha : X \times Y \to Z \text{ is } C^0.$$

Conversely, if α is C^0,

$\beta(x) : Y \to Z$ can be written as composite

$$Y \longrightarrow X \times Y \xrightarrow{\alpha} Z$$
$$y \longmapsto (x, y)$$

so each $\beta(x)$ is C^0.

Further, β is the composition

$$X \xrightarrow{inj} \mathrm{Maps}(Y, X \times Y) \xrightarrow{\alpha_*} \mathrm{Maps}(Y, Z)$$

which is continuous. //

Cor Let W be any space. Then the sets

$$\mathrm{Maps}(W, \mathrm{Maps}(X, \mathrm{Maps}(Y, Z)))$$

and

$$\mathrm{Maps}(W, \mathrm{Maps}(X \times Y, Z))$$

are naturally isomorphic.

Pf

$$\begin{array}{c} \mathrm{Maps}(W, \mathrm{Maps}(X, \mathrm{Maps}(Y,Z))) \\ \cong \\ \mathrm{Maps}(W \times X, \mathrm{Maps}(Y, Z)) \\ \cong \\ \mathrm{Maps}(W \times X \times Y, Z) \\ \cong \\ \mathrm{Maps}(W, \mathrm{Maps}(X \times Y, Z)) \end{array}$$

as sets, by Lemma (c). //

That is, $\left.\begin{array}{l} \mathrm{Maps}(-, \mathrm{Maps}(X, \mathrm{Maps}(Y, Z))) \\ \mathrm{Maps}(-, \mathrm{Maps}(X \times Y, Z)) \end{array}\right. : \mathrm{Spaces}^{op} \longrightarrow \mathrm{Sets}$

are naturally isomorphic functors.

Thm (Yoneda Lemma).

Given any category $\mathcal{C}$, if

$$\hom(-, A)$$

and

$$\hom(-, B)$$

are naturally isomorphic functors, then

$$A \cong B \quad \text{in } \mathcal{C}.$$

Cor The spaces

$$\mathrm{Maps}(X, \mathrm{Maps}(Y, Z))$$

and

$$\mathrm{Maps}(X \times Y, Z)$$

are naturally homeomorphic whenever X, Y, Z are k-spaces.

Why do we also want weak Hausdorff? For certain quotients.

Prop (Homework)

If X is a k-space
Y is compact generated, then $\mathrm{Maps}(X, Y)$ is compactly generated.