

Limits and Colimits

Things like quotients, unions, products, are limits and colimits.
Quotients of what? Products of what? The "what" is specified by a diagram.

Defn let \mathcal{C} and \mathcal{D} be categories.

Then a \mathcal{D} -shaped diagram in \mathcal{C}

is a functor

$$\mathcal{D} \xrightarrow{D} \mathcal{C}.$$

Ex.

• If \mathcal{D} is a category w/ a single object, and a single morphism (id) $\bullet \rightarrow \bullet = \text{id}$ then a \mathcal{D} -shaped diagram is just an object of \mathcal{C} .

• If \mathcal{D} is a category $\bullet \rightarrow \ast = \mathcal{D}$ w/ two objects, and a single non-identity morphism, a \mathcal{D} -shaped diagram is a choice of two objects and a morphism between them.

• $\begin{matrix} \mathcal{D} & \rightarrow & \mathcal{D} \\ \downarrow & & \\ \mathcal{D} & & \end{matrix} = \mathcal{D}$, a \mathcal{D} -shaped diagram is $\begin{matrix} \mathcal{D}_0 & \xrightarrow{f_01} & \mathcal{D}_1 \\ \downarrow f_02 & & \\ \mathcal{D}_2 & & \end{matrix}$ in \mathcal{C} .

• $\mathcal{D} \rightarrow \mathcal{D}_1 \rightarrow \mathcal{D}_2 \rightarrow \dots = \mathcal{D}$, then $\mathcal{D}_0 \rightarrow \mathcal{D}_1 \rightarrow \mathcal{D}_2 \rightarrow \dots$ in \mathcal{C} .

• If the only morphisms of \mathcal{D} are id, then a \mathcal{D} -shaped diagram is a choice of collection of objects of \mathcal{C} .

Ex For any \mathcal{D} , and for any $X \in \mathcal{C}$,
the constant functor

$$\begin{aligned} \mathcal{D} &\longrightarrow \mathcal{C} \\ \text{ob } \mathcal{D} &\longmapsto \{X\} \in \text{ob } \mathcal{C} \\ \text{hom } \mathcal{D} &\longrightarrow \{id_X\} \end{aligned}$$

is a \mathcal{D} -shaped diagram.

Def (Recall) A natural transformation from F_0 to F_1
~~is~~ (two functors $F_0, F_1: \mathcal{D} \rightarrow \mathcal{C}$)
is a choice of morphism

$$\eta_{D,i}: F_0(D_i) \rightarrow F_1(D_i) \in \text{hom}_{\mathcal{C}}(F_0(D_i), F_1(D_i)), \forall D_i \in \text{ob } \mathcal{D}$$

s.t. $\forall f: D_i \xrightarrow{j} D_j$ in \mathcal{D} ,

$$\begin{array}{ccc} F_0(D_i) & \xrightarrow{\eta_{D,i}} & F_1(D_i) \\ F_0(f) \downarrow j & & \downarrow F_1(f) \\ F_0(D_j) & \xrightarrow{\eta_{D,j}} & F_1(D_j) \end{array}$$

commutes.

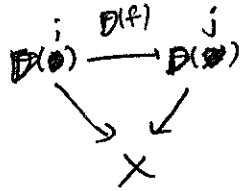
Ex

\mathcal{D}	$F: \mathcal{D} \rightarrow \mathcal{C}$	η (natural trans)
\bullet	$F(\bullet) = X \in \text{ob } \mathcal{C}$	$D_0 \xrightarrow{\eta} D_1$ " $F(X)$ \downarrow $F(X)$
$\mathcal{A} \rightarrow \mathcal{B}$	$F(\mathcal{A}) \xrightarrow{f} F(\mathcal{B})$	$F_0(\mathcal{A}) \xrightarrow{\eta_{\mathcal{A}}} F_0(\mathcal{B})$ $f_0 \downarrow \quad \quad \downarrow f_1$ $F_0(\mathcal{A}) \xrightarrow{\eta_{\mathcal{A}}} F_0(\mathcal{B})$

choice of morphism in \mathcal{C}

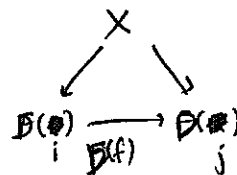
choice of $\eta_{\mathcal{A}}, \eta_{\mathcal{B}}$
making this commute

Ex If $\mathcal{D} \xrightarrow{X} \mathcal{C}$ is the constant diagram w/ value X , a natural trans. from \mathcal{D} to X is a choice of morphism $\mathcal{D}(i) \rightarrow X \forall i \in \text{ob } \mathcal{D}$, s.t.



commutes $\forall f \in \text{hom}_{\mathcal{D}}(i, j)$.

From X to \mathcal{D} is a choice of morphism $X \rightarrow \mathcal{D}(i) \forall i \in \text{ob } \mathcal{D}$ s.t.



commutes $\forall f \in \text{hom}_{\mathcal{D}}(i, j)$.

Ex Let \mathcal{D} be a category where any homs are id.

Then a functor $\mathcal{D} \rightarrow \mathcal{C}$ is a choice of object $X_i \forall i \in \text{ob } \mathcal{D}$.

If $\mathcal{C} = \text{Spaces}$, a choice of space $X_i \forall i \in \text{ob } \mathcal{D}$.

Let $X = \prod_{\text{ob } \mathcal{D}} X_i$. Then \exists nat. trans

$$\underline{X} \longrightarrow \mathcal{D}$$

by setting

$$\eta_{i_0}: \prod X_{i_1} \rightarrow X_{i_0}$$

Likewise, if $Y = \coprod_{\text{ob } \mathcal{D}} X_i$, \exists nat. trans

$$F \longrightarrow \underline{Y}$$

by setting

$$\eta_{i_0}: X_{i_0} \rightarrow \coprod_{\text{ob } \mathcal{D}} X_i$$

Ex If $\mathcal{D} = \{ a \rightarrow b \}$ is category of three objects,

$0 < 1$
 $0 < 1'$
 $1 \times 1'$
 $1' \times 1$
not
related

and ~~$\text{hom}(a,b) = \{*\}$~~
 ~~$\text{hom}(a,c) = \{*\}$~~
 ~~$\text{hom}(b,c) = \{*\}$~~
 ~~$\text{hom}(c,a) = \{*\}$~~
 ~~$\text{hom}(b,a) = \{*\}$~~
 ~~$\text{hom}(c,b) = \{*\}$~~
 $\text{hom}(i,j) = \begin{cases} * & \text{if } i=j \\ \emptyset & \text{otherwise} \end{cases}$

Then $\mathcal{D}: \mathcal{D} \rightarrow \text{Spaces}$ is just a choice of ~~map~~ spaces ~~X_a, X_b, X_c~~ D_0, D_1, D_2
and morphisms $D_0 \xrightarrow{f} D_1, D_0 \xrightarrow{g} D_2$.

Let $X = D_0 \amalg D_1 \amalg D_2 / \begin{matrix} x \sim f(x) \\ x \sim g(x) \end{matrix} \forall x \in D_0$.

Then \exists nat. trans.

$$\mathcal{D} \rightarrow X$$

i.e., a commutative diagram

$$\begin{array}{ccc} D_a & \xrightarrow{f} & D_b \\ g \downarrow & & \downarrow h_b \\ D_c & \longrightarrow & X \end{array} = \begin{array}{ccc} D_a & \xrightarrow{f} & D_b \\ & \searrow h_a & \downarrow h_b \\ D_c & \xrightarrow{h_c} & X \end{array}$$

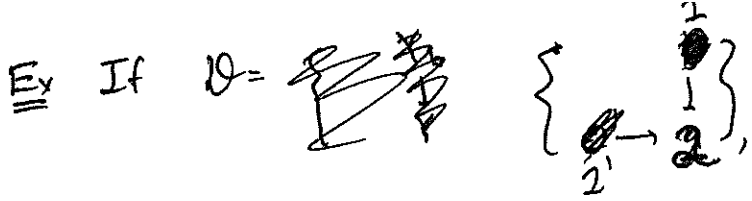
where $h_a(x) = [x] \in X$, etc.

Likewise, if $X = *$, \exists nat. trans.

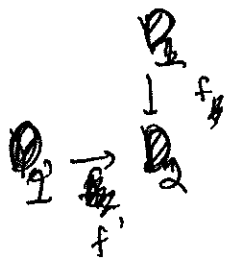
$$\mathcal{D} \rightarrow X$$

given by

$$\begin{array}{ccc} D_a & \rightarrow & D_b \\ \downarrow & & \downarrow \\ D_c & \rightarrow & * \end{array}$$

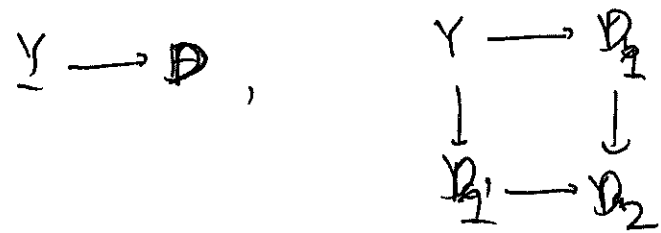


Then $\mathcal{D}: \mathcal{D} \rightarrow \text{Spaces}$ is a choice

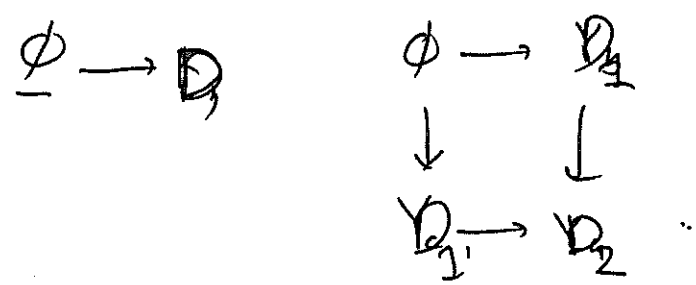


Let $Y = \{ (y, q) \in \mathcal{D}_1 \times \mathcal{D}_2 \mid f_*(y) = f'_*(q) \} \subset \mathcal{D}_1 \times \mathcal{D}_2$

Then \exists nat. trans



Likewise, \exists nat. trans



Def. Fix $D: \mathcal{D} \rightarrow \mathcal{C}$, and an object $X \in \text{ob } \mathcal{C}$.

A map from D to X is a natural transformation

$$\eta: D \rightarrow \underline{X}.$$

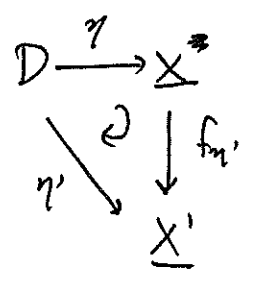
A map from X to D is a nat. tra.

$$\eta: \underline{X} \rightarrow D.$$

Def. A map from D to X is called initial if

\forall other maps $D \xrightarrow{\eta'} \underline{X'}$, $\exists!$ morphism $f_{\eta}: X \rightarrow X'$

s.t.



commutes.

A colimit for D is an initial map from D to X.

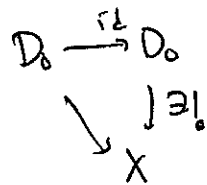
Ex If $\mathcal{D} = \{\emptyset\}$ then

$$D: \mathcal{D} \rightarrow \text{Spaces}$$

is a choice of space D_0 .

Any $\eta: D_0 \rightarrow X$ is a choice of cts map $D_0 \rightarrow X$.

Well, any map $D_0 \rightarrow X$ factors thugh the map



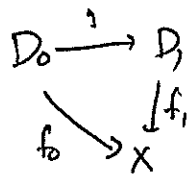
So $D \xrightarrow{\eta} \underline{D_0}$, $\eta: D_0 \xrightarrow{\text{id}} D_0$ is the colimit.

Ex If $\mathcal{D} = 0 \rightarrow 1$, then

$$D: \mathcal{D} \rightarrow \text{Spaces}$$

is a choice of cts map $D_0 \xrightarrow{g} D_1$.

Any $\eta: D \rightarrow X$ is a choice $f_i: D_i \rightarrow X$ s.t.



commutes

Any such diagram factors thugh $D_0 \xrightarrow{g} D_1$

$$\begin{array}{ccc}
 D_0 & \xrightarrow{g} & D_1 \\
 & \searrow & \downarrow \text{id} \\
 & & D_1
 \end{array}$$

via
$$\begin{array}{ccccc}
 D_0 & \xrightarrow{g} & D_1 & & \\
 \searrow f_0 & \downarrow g & \downarrow \text{id} & \searrow f_1 & \\
 & D_1 & \downarrow \text{id} & & \\
 & \searrow & X & &
 \end{array}$$
, and uniquely. So colim $D = D_1$.

Ex Let $\mathcal{D} = \begin{array}{ccc} 0 & \longrightarrow & 1 \\ & & \downarrow \\ & & 1' \end{array}$

and $D: \mathcal{D} \rightarrow \text{Spaces}$ s.t.

$$D(\mathcal{D}) = \begin{array}{ccc} D_0 & \xrightarrow{g} & D_1 \\ & & \downarrow \\ & & D_1 = * \end{array}$$

Then

$$X = D_1 / g(D_0) = \cancel{D_1} // \cancel{D_1} / \sim \forall x \in \text{map}(\mathcal{D})$$

receives a map:

$$\begin{array}{ccc} D_0 & \longrightarrow & D_1 \\ \downarrow & & \downarrow \\ * & \longrightarrow & X \end{array}$$

X is colimit, since any map

$$\begin{array}{ccc} D_0 & \longrightarrow & D_1 \\ \downarrow & & \searrow \\ * & & X' \end{array}$$

[i.e., quotients are colimits]

must factor thgh X .

$$\begin{array}{ccc} D_0 & \longrightarrow & D_1 \\ \downarrow & & \downarrow \\ * & \longrightarrow & X \end{array} \begin{array}{c} \searrow \\ \xrightarrow{\exists! \epsilon} \\ \searrow \end{array} \begin{array}{ccc} & & X' \\ & \xrightarrow{\epsilon} & \\ & \searrow & \end{array}$$

Defn A map

$$\underline{X} \xrightarrow{h} D$$

is called terminal if \forall other nat. trans.

$$\underline{X}' \xrightarrow{h'} D,$$

$\exists!$ morphism $\underline{X}' \rightarrow \underline{X}$ s.t.

$$\begin{array}{ccc} \underline{X}' & \longrightarrow & \underline{X} \\ & \searrow h' & \downarrow h \\ & & D \end{array}$$

commutes.

Given $D: \mathcal{D} \rightarrow \mathcal{C}$, a terminal map $\underline{X} \rightarrow D$ is called a limit for D .

Ex. If $\mathcal{D} = \{0 \circlearrowleft \text{id}_0\}$, $D: \mathcal{D} \rightarrow \text{Spaces}$

is a choice of space D_0 , and a limit is $\underline{D}_0 \xrightarrow{\text{id}} D$.

• If $\mathcal{D} = \{0 \rightarrow 1\}$, $D: \mathcal{D} \rightarrow \text{Spaces}$ a choice $D_0 \xrightarrow{g} D_1$,

and a limit is $\underline{D}_0 \xrightarrow{(\text{id}, g)} D$.

$$= \begin{array}{ccc} & D_0 & \\ \text{id} \swarrow & & \searrow g \\ D_0 & \xrightarrow{g} & D_1 \end{array}$$

Ex If $D = \left\{ \begin{array}{c} 1 \\ 1 \rightarrow \frac{1}{2} \end{array} \right\}$,

fix $D: D \rightarrow \mathbb{C}$ s.t. $D_3 = *$.

$$\begin{array}{ccc} & D_1 & \\ & \downarrow g & \\ * & \xrightarrow{x_0} & D_2 \end{array}, \quad x \mapsto x_0 \in D_2.$$

Then a limit for D is

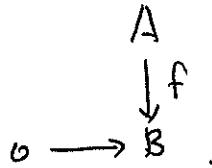
X , where $X = \{ \cancel{x \in D_1} \mid g(x) = x_0 \}$.

$$\begin{array}{ccc} X & \hookrightarrow & D_1 \\ \downarrow & & \downarrow g \\ * & \xrightarrow{x_0} & D_2 \end{array}$$

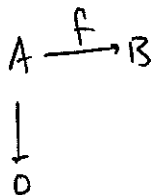
Exer Let $\mathcal{C} = \text{Abgps}$.

Fix a map $A \xrightarrow{f} B$.

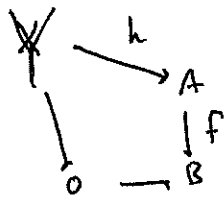
(1) Find the limit of



(2) Find the colimit of

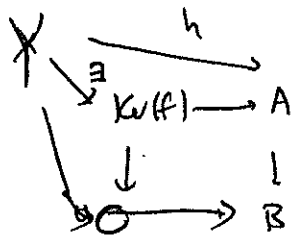


Ans: (1)



commutes iff $h(\ker f) \subseteq \{a \mid f(a) = 0\} = \ker(f)$.

So h factors through $\ker(f)$.



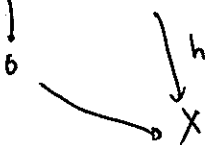
This factorization is unique b/c

$$\ker(f) \hookrightarrow A$$

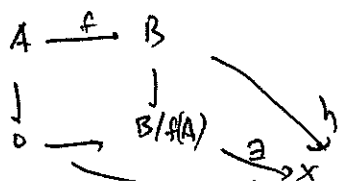
is an injection.

(2) $A \xrightarrow{f} B$

commutes iff $\text{im}(f) \subseteq \ker(h)$ i.e., if $h \circ f(A) = 0$.



So h descends to a map $B/f(A) = \text{Colim}(f)$.



This factorization is unique since $B \twoheadrightarrow B/f(A)$ is a surjection.