Limits and Colimits

Things like quotients, unions, products, are limits and colimits. Quotients of what? Products of what? The "what" is specified by a diagram.

Defn Let $\mathcal{C}$ and $\mathcal{D}$ be categories.

Then a __$\mathcal{D}$-shaped diagram__ in $\mathcal{C}$ is a functor

$$\mathcal{D} \xrightarrow{D} \mathcal{C}.$$

Ex.

- If $\mathcal{D}$ is a category w/ a single object, and a single morphism (id) $\bullet = \mathcal{D}$ then a $\mathcal{D}$-shaped diagram is just an object of $\mathcal{C}$.

- If $\mathcal{D}$ is a category $\bullet \longrightarrow * = \mathcal{D}$ w/ two objects, and a single non-identity morphism, a $\mathcal{D}$-shaped diagram is a choice of two objects and a morphism between them.

- $$\begin{array}{ccc} 0 & \rightarrow & 1 \\ \downarrow & & \\ 2 & & \end{array} = \mathcal{D}$$, a $\mathcal{D}$-shaped diagram is $$\begin{array}{ccc} D_0 & \xrightarrow{f_{01}} & D_1 \\ {\scriptstyle f_{02}}\downarrow & & \\ D_2 & & \end{array}$$ in $\mathcal{C}$.

- $0 \rightarrow 1 \rightarrow 2 \rightarrow \cdots = \mathcal{D}$, then $D_0 \rightarrow D_1 \rightarrow D_2 \rightarrow \cdots$ in $\mathcal{C}$.

- If the only morphisms of $\mathcal{D}$ are id, then a $\mathcal{D}$-shaped diagram is a choice of collection of objects of $\mathcal{C}$.

Ex For any $\mathcal{D}$, and for any $X \in \mathcal{C}$,

the constant functor

$$\mathcal{D} \longrightarrow \mathcal{C}$$

$$\mathrm{ob}\,\mathcal{D} \longmapsto \{X\} \in \mathrm{ob}\,\mathcal{C}$$

$$\mathrm{hom}\,\mathcal{D} \longrightarrow \{id_X\}$$

is a $\mathcal{D}$-shaped diagram.

Defn (Recall) A natural transformation from F_0 to F_1

(two functors $F_0, F_1 : \mathcal{D} \to \mathcal{C}$)

is a choice of morphism

$$\eta_i : F_0(i) \longrightarrow F_1(i) \quad \in \mathrm{hom}_{\mathcal{C}}(F_0(i), F_1(i)), \ \forall\, i \in \mathrm{ob}\,\mathcal{D}$$

s.t. $\forall \quad f : i \to j$ in $\mathcal{D}$,

$$\begin{tikzcd}
F_0(i) \arrow[r, "\eta_i"] \arrow[d, "F_0(f)"'] & F_1(i) \arrow[d, "F_1(f)"] \\
F_0(j) \arrow[r, "\eta_j"'] & F_1(j)
\end{tikzcd}$$

commutes.

Ex

$\mathcal{D}$	$F : \mathcal{D} \to \mathcal{C}$	η (natural trans)	
$\bullet$	$F(\bullet) = X \in \mathrm{ob}\,\mathcal{C}$	$\begin{array}{ccc} X_0 & \xrightarrow{\eta} & X_1 \\ \parallel & & \parallel \\ F(X_0) & & F(X_1) \end{array}$	choice of morphism in $\mathcal{C}$
$a \to b$	$F(a) \xrightarrow{f} F(b)$	$\begin{array}{ccc} F_0(a) & \xrightarrow{\eta_a} & F_1(a) \\ {\scriptstyle f_0}\downarrow & \circlearrowleft & \downarrow{\scriptstyle f_1} \\ F_0(b) & \xrightarrow[\eta_b]{} & F_1(b) \end{array}$	choice of η_a, η_b making this commute

<u>Ex</u> If $\underline{X}: \mathcal{D} \to \mathcal{C}$ is the constant diagram w/ value X,

a natural trans. from D to $\underline{X}$ is a choice of morphism $D(i) \to X$ $\forall\, i \in ob\,\mathcal{D}$, s.t.

$$\begin{tikzcd} D(i) \arrow[rr, "D(f)"] \arrow[dr] & & D(j) \arrow[dl] \\ & X & \end{tikzcd}$$

commutes $\forall\, f \in hom_{\mathcal{D}}(i,j)$.

from $\underline{X}$ to D is a choice of morphism $X \to D(i)$ $\forall\, i \in ob\,\mathcal{D}$

s.t.

$$\begin{tikzcd} & X \arrow[dl] \arrow[dr] & \\ D(i) \arrow[rr, "D(f)"'] & & D(j) \end{tikzcd}$$

commutes $\forall\, f \in hom_{\mathcal{D}}(i,j)$.

<u>Ex</u> Let $\mathcal{D}$ be a category where only homs are id.

Then a functor $D: \mathcal{D} \to \mathcal{C}$ is a choice of object X_i $\forall\, i \in ob\,\mathcal{D}$.

If $\mathcal{C} =$ Spaces, a choice of space X_i $\forall\, i \in ob\,\mathcal{D}$.

Let $X = \prod_{ob\,D} X_i$. Then $\exists$ nat. trans

$$\underline{X} \longrightarrow D$$

by setting

$$\eta_{i_0}: \prod X_i \longrightarrow X_{i_0}$$

Likewise, if $Y = \coprod_{ob\,\mathcal{D}} X_i$, $\exists$ nat. trans

$$F \longrightarrow \underline{Y}$$

by setting

$$\eta_{i_0}: X_{i_0} \longrightarrow \coprod_{ob\,D} X_i .$$

Ex If $\mathcal{D} = \left\{ \begin{tikzcd}[cramped] 0 \arrow[r] \arrow[d] & 1 \\ 1' \end{tikzcd} \right\}$ is category w/ three objects,

$0 < 1$
$0 < 1'$
$1 \not< 1'$
$1' \not< 1$
not related

and ~~$hom(0,1) = \{*\}$~~ $\quad hom(i,j) = \begin{cases} * & \text{if } i \leq j \\ \phi & \text{otherwise} \end{cases}$

~~$hom(0,1') = \{*\}$~~

~~$hom(a,a) = \{id_a\}$, $hom(b,b) = \{id_b\}$, $hom(c,c) = \{id_c\}$~~

~~$hom(b,a) = hom(c,a) = hom(c,b) = hom(b,c) = \{\phi\}$.~~

Then $D : \mathcal{D} \longrightarrow$ Spaces is just a choice of ~~maps~~ spaces ~~X_a, X_b, X_c~~ D_0, D_1, D_2, and morphisms $D_0 \xrightarrow{f} D_1$, $D_0 \xrightarrow{g} D_{1'}$).

Let $\quad X = D_1 \amalg D_{1'} \amalg D_0 \Big/ \begin{matrix} x \sim f(x) \\ x \sim g(x) \end{matrix} \; \forall \, x \in D_0.$

Then $\exists$ nat. trans.

$$D \longrightarrow \underline{X}$$

i.e., a commutative diagram

$$\begin{tikzcd} D_a \arrow[r, "f"] \arrow[d, "g"'] & D_b \arrow[d] \\ D_c \arrow[r] & X \end{tikzcd} \quad = \quad \begin{tikzcd} D_a \arrow[r, "f"] \arrow[d, "g"'] \arrow[dr, "h_a"] & D_b \arrow[d, "h_b"] \\ D_c \arrow[r, "h_c"'] & X \end{tikzcd}$$

where $\quad h_a(x) = [x] \in X$, etc.

Likewise, if $X = *$, $\exists$ nat. trans.

$$D \longrightarrow \underline{X}$$

given by

$$\begin{tikzcd} D_a \arrow[r] \arrow[d] & D_b \arrow[d] \\ D_c \arrow[r] & * \end{tikzcd}.$$

Ex If $\mathcal{D} = \left\{ \begin{tikzcd} & 1 \arrow[d] \\ 2' \arrow[r] & 2 \end{tikzcd} \right\}$,

then $D : \mathcal{D} \to \text{Spaces}$ is a choice

$$\begin{tikzcd} & D_1 \arrow[d, "f_*"] \\ D_{1'} \arrow[r, "f'_*"'] & D_2 \end{tikzcd}.$$

Let

$$Y = \{ (y, y') \in D_1 \times D_{1'} \mid f_*(y) = f'_*(y') \} \subset D_1 \times D_{1'},$$

Then $\exists$ nat. trans

$$\underline{Y} \to D, \qquad \begin{tikzcd} Y \arrow[r] \arrow[d] & D_1 \arrow[d] \\ D_{1'} \arrow[r] & D_2 \end{tikzcd}.$$

Likewise, $\exists$ nat trans,

$$\underline{\emptyset} \to D, \qquad \begin{tikzcd} \emptyset \arrow[r] \arrow[d] & D_1 \arrow[d] \\ D_{1'} \arrow[r] & D_2 \end{tikzcd}.$$

<u>Defn.</u> Fix $D : \mathcal{D} \to \mathcal{C}$, and an object $X \in \mathrm{ob}\,\mathcal{C}$.

A <u>map</u> <u>from</u> <u>D to X</u> is a natural transformation

$$\eta : D \to \underline{X}.$$

A map from X to D is a nat. trans

$$\eta : \underline{X} \to D.$$

<u>Defn</u> A map from D to X is called <u>initial</u> if

$\forall$ other maps $D \xrightarrow{\eta'} \underline{X}'$, $\exists!$ morphism $f_{\eta'} : X \to X'$

s.t.

$$\begin{tikzcd} D \arrow[r, "\eta"] \arrow[dr, "\eta'"'] & \underline{X} \arrow[d, "f_{\eta'}"] \arrow[dl, phantom, "\circlearrowleft"] \\ & \underline{X'} \end{tikzcd}$$

commutes.

A <u>colimit</u> for D is an initial map from D to X.

Ex If $\mathcal{D} = \{0\}$ then

$$D : \mathcal{D} \to \text{Spaces}$$

is a choice of space D_0.

Any $\eta : D \to \underline{X}$ is a choice of cts map $D_0 \to X$.

Well, any map $D_0 \to X$ factors thgh the map

$$\begin{tikzcd} D_0 \arrow[r, "id"] \arrow[dr] & D_0 \arrow[d, "\exists!"] \\ & X \end{tikzcd}$$

So $D \xrightarrow{\eta} \underline{D_0}$, $\eta : D_0 \xrightarrow{id} D_0$ is the colimit.

Ex If $\mathcal{D} = 0 \to 1$, then

$$D : \mathcal{D} \to \text{Spaces}$$

is a choice of cts map $D_0 \xrightarrow{g} D_1$.

Any $\eta : D \to \underline{X}$ is a choice $f_i : D_i \to X$ s.t.

$$\begin{tikzcd} D_0 \arrow[r, "g"] \arrow[dr, "f_0"'] & D_1 \arrow[d, "f_1"] \\ & X \end{tikzcd}$$

commutes.

Any such diagram factors thgh
$$\begin{tikzcd} D_0 \arrow[r, "g"] \arrow[dr, "g"'] & D_1 \arrow[d, "id"] \\ & D_1 \end{tikzcd}$$

via
$$\begin{tikzcd} D_0 \arrow[rr, "g"] \arrow[dr, "g"] \arrow[ddr, "f_0"'] & & D_1 \arrow[dl, "id"'] \arrow[ddl, "f_1"] \\ & D_1 \arrow[d, "f_1"] & \\ & X & \end{tikzcd}$$
, and uniquely. So colim $D = D_1$.

Ex Let $\mathcal{D} =$
$$\begin{tikzcd} 0 \arrow[r] \arrow[d] & 1 \\ 1' & \end{tikzcd}$$

and $D : \mathcal{D} \longrightarrow \mathcal{S}paces$ s.t.

$$D(\mathcal{D}) = \begin{tikzcd} D_0 \arrow[r, "g"] \arrow[d] & D_1 \\ D_{1'} = * & \end{tikzcd}.$$

Then
$$X = D_1 / g(D_0) = * \amalg D_1 \Big/ \begin{array}{l} * \sim x \\ \forall\, x \in \operatorname{image}(D) \end{array}$$

receives a map:

$$\begin{tikzcd} D_0 \arrow[r] \arrow[d] & D_1 \arrow[d] \\ * \arrow[r] & X \end{tikzcd}.$$

X is colimit, since any map

$$\begin{tikzcd} D_0 \arrow[r] \arrow[d] & D_1 \arrow[ddr, bend left] & \\ * \arrow[drr, bend right] & & \\ & & X' \end{tikzcd}$$

[i.e., quotients are colimits]

must factor thru X.

$$\begin{tikzcd} D_0 \arrow[r] \arrow[d] & D_1 \arrow[d] \arrow[ddr, bend left] & \\ * \arrow[r] \arrow[drr, bend right] & X \arrow[dr, dashed, "\exists!"] & \\ & & X' \end{tikzcd}$$

<u>Defn</u> A map

$$\underline{X} \xrightarrow{h} D$$

is called <u>terminal</u> if $\forall$ other nat. trans.

$$\underline{X}' \xrightarrow{h'} D,$$

$\exists!$ morphism $X' \longrightarrow X$ s.t.

$$\begin{tikzcd} \underline{X}' \arrow[r] \arrow[dr, "h'"'] & \underline{X} \arrow[d, "h"] \\ & D \end{tikzcd}$$

commutes.

Given $D : \mathcal{D} \to \mathcal{C}$, a terminal map $\underline{X} \xrightarrow{h} D$ is called a <u>limit</u> for D.

<u>Ex</u> . If $\mathcal{D} = \{0 \circlearrowleft^{id_0}\}$, $D : \mathcal{D} \to$ Spaces is a choice of space D_0, and a limit is $\underline{D_0} \xrightarrow{id} D$.

. If $\mathcal{D} = \{0 \to 1\}$, $D : \mathcal{D} \to$ Spaces is a choice $D_0 \xrightarrow{g} D_1$, and a limit is $\underline{D_0} \xrightarrow{(id, g)} D$. =

$$\begin{tikzcd} & D_0 \arrow[dl, "id"'] \arrow[dr, "g"] & \\ D_0 \arrow[rr, "g"] & & D_1 \end{tikzcd}$$

Ex If $\mathcal{D} = \left\{ \begin{tikzcd} & 1 \arrow[d] \\ 1' \arrow[r] & 2 \end{tikzcd} \right\}$,

fix $D : \mathcal{D} \to \mathcal{C}$ s.t. $D_{1'} = *$.

$$\begin{tikzcd} & D_1 \arrow[d, "g"] \\ * \arrow[r, "x_0"'] & D_2 \end{tikzcd} , \quad * \mapsto x_0 \in D_2 .$$

Then a limit for D is

$\underline{X}$, where $X = \{ x \in D_1 \mid g(x) = x_0 \}$.

$$\begin{tikzcd} X \arrow[r, hook] \arrow[d] & D_1 \arrow[d, "g"] \\ * \arrow[r, "x_0"'] & D_2 \end{tikzcd} .$$

<u>Exer</u> Let $\mathcal{C} =$ Abgps.

Fix a map $A \xrightarrow{f} B$.

(1) Find the limit of

$$\begin{tikzcd} & A \arrow[d, "f"] \\ 0 \arrow[r] & B \end{tikzcd}$$

(2) Find the colimit of

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d] & B \\ 0 & \end{tikzcd}$$

<u>Ans</u>: (1)

$$\begin{tikzcd} Y \arrow[r, "h"] \arrow[d] & A \arrow[d, "f"] \\ 0 \arrow[r] & B \end{tikzcd}$$

commutes iff $h(Y) \subset \{a \mid f(a) = 0\}$
$= \mathrm{Ker}(f)$.

So h factors thgh $\mathrm{Ker}(f)$.

$$\begin{tikzcd} Y \arrow[dr, "\exists"] \arrow[drr, bend left, "h"] \arrow[ddr, bend right] & & \\ & \mathrm{Ker}(f) \arrow[r] \arrow[d] & A \arrow[d] \\ & 0 \arrow[r] & B \end{tikzcd}$$

This factoring is unique b/c
$\mathrm{Ker}(f) \hookrightarrow A$
is an injection.

(2)

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d] & B \arrow[dr, "h"] & \\ 0 \arrow[rr, bend right] & & X \end{tikzcd}$$

commutes iff $\mathrm{im}(f) \subset \mathrm{Ker}(h)$ i.e., if
$h \circ f(A) = 0$.

So h descends to a map $B/f(A) = \mathrm{Coker}(f)$.

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d] & B \arrow[d] \arrow[dr, bend left, "h"] & \\ 0 \arrow[r] \arrow[rr, bend right] & B/f(A) \arrow[r, "\exists"] & X \end{tikzcd}$$

This factoring is unique since $B \longrightarrow B/f(A)$
is a surjection.