~~Prop~~ <u>Defn</u>. Let $A, B \in \mathrm{ob}\,\mathcal{C}$.
A <u>coproduct</u> of A and B is a colimit to the diagram given by A and B.

<u>Ex</u>

- $\mathcal{C}$ = Groups. Then coproduct of A and B is the free group

$$A * B$$

- $\mathcal{C}$ = Abelian Groups. The coproduct is the direct sum

$$A \oplus B$$

$$\begin{tikzcd} & A \arrow[d] \arrow[ddr, bend left] & \\ B \arrow[r] \arrow[drr, bend right] & A \oplus B \arrow[dr, dashed, "\exists!"'] & \\ & & X \end{tikzcd}$$

- $\mathcal{C}$ = Spaces.

$$A \amalg B$$

- $\mathcal{C}$ = Spaces_*

$$A \vee B.$$

- $\mathcal{C}$ = Sets

$$A \amalg B$$

Defn Let $A, B \in \mathrm{ob}\, \mathcal{C}$.
A *product*, or *direct product*, of A and B is a limit to the diagram given by A and B.

$$
\begin{tikzcd}
Y \arrow[dr, dashed, "\exists!"] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\
& A \times B \arrow[r] \arrow[d] & A \\
& B &
\end{tikzcd}
$$

Ex
- $\mathcal{C}$ = Groups

  $A \times B$

- $\mathcal{C}$ = Abelian grps

  $A \times B \cong A \oplus B$

- $\mathcal{C}$ = Spaces

  $A \times B$

- $\mathcal{C}$ = Spaces_*

  $A \times B$

- $\mathcal{C}$ = Sets

  $A \times B$

Let

$$\widetilde{H}_n : \text{Spaces}_* \longrightarrow \text{AbGrps}$$

$$\pi_n : \text{Spaces}_* \longrightarrow \text{Grps}$$

be the usual functors.

<u>Prop</u> Let (X, x_0) and (Y, y_0) be ptd spaces s.t. each basepoint has a contractible open neighborhood. Then

$$\widetilde{H}_n(X \vee Y) \cong \widetilde{H}_n(X) \oplus \widetilde{H}_n(Y) \quad \forall\, n,$$

and

$$\pi_1(X \vee Y) \cong \pi_1(X) * \pi_1(Y).$$

Also,

$$\pi_n(X \times Y) \cong \pi_n(X) \times \pi_n(Y) \quad \forall\, n.$$

In other words, for reasonable spaces, $\widetilde{H}_n$ preserves coproducts, and π_n preserves products. That π_1 preserves coproducts is particular to π_1.

The Hurewicz Theorem

There's a map

$$\pi_n(X, x_0) \longrightarrow \underset{\substack{\text{\rotatebox{90}{$\cong$}} \\ H_n(X, x_0)}}{} \tilde{H}_n(X), \quad n \geq 1.$$

For $n=0$, there's still a map, but only of sets.

for any space (X, x_0) (pointed).

If $g: (S^n, *) \longrightarrow (X, x_0)$

represents an element of π_n, it induces a map on homology

$$g_* : \underset{\mathbb{Z}}{\tilde{H}_n(S^n)} \longrightarrow \tilde{H}_n(X) \cong H_n(X, x_0)$$

Fixing a generator $1 \in \tilde{H}_n(S^n)$; g_* picks out an element of $\tilde{H}_n(X)$. Since maps on H_* are unaffected by homotopies of maps, the map

$\pi_n(X, x_0) \longrightarrow \tilde{H}_n(X)$ is well-defined

<u>Prop'n</u> The Hurewicz map defines a natural transformation from π_n to $\widetilde{H}_n$, $\forall\ n \geq 1$.

Explicitly, $\pi_n : \text{Spaces}_* \longrightarrow \text{Groups}$

$\widetilde{H}_n : \text{Spaces}_* \longrightarrow \text{Groups}$ $\quad n \geq 1$

are the two usual functors, (which ~~the~~ happen to factor through the category of abelian groups if $n \geq 2$) and $\forall\ f : (X, x_0) \longrightarrow (Y, y_0)$, we have a commutative diagram

$$\begin{tikzcd}
\pi_n(X, x_0) \arrow[r, "f_*"] \arrow[d] & \pi_n(Y, y_0) \arrow[d] \\
\widetilde{H}_n(X, x_0) \arrow[r, "f_*"] & \widetilde{H}_n(Y, y_0)
\end{tikzcd}$$

Pf. Commutativity is obvious as a map of sets:

$$
\begin{tikzcd}
{[g]} \arrow[r, mapsto] \arrow[d, mapsto] & {[f\circ g]} \arrow[d, mapsto] \\
 & (f\circ g)_*(1) \\
g_*(1) \arrow[r, mapsto] & f_* g_*(1)
\end{tikzcd}
$$

, $1 \in H_n(S_n, *)$ fixed generator.

We just need to prove $[g] \mapsto g_*(1)$ is a group homomorphism.

$\forall$ maps $g: A \to X$
$g': A' \to X$,

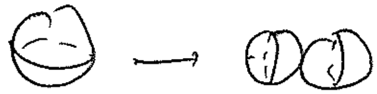
Consider the commutative diagrams

$$
\begin{tikzcd}
 & A \arrow[d, "i"] \arrow[ddr, "g"] & & & & \widetilde{H}_n(A) \arrow[d, "i_*"] \arrow[ddr, "g_*"] & \\
A' \arrow[r, "i'"] \arrow[drr, bend right, "g'"'] & A\vee A' \arrow[dr, "g\vee g'"] & \arrow[r, "\widetilde{H}_n"] & {} & \widetilde{H}_n(A') \arrow[r] \arrow[drr, bend right] & \widetilde{H}_n(A)\oplus\widetilde{H}_n(A') \arrow[dr] & \\
 & & X & & & & \widetilde{H}_n(X)
\end{tikzcd}
$$

Since $\widetilde{H}_\circ$ preserves coproducts, the map $A\vee A' \xrightarrow{g\vee g'} X$ induces the map $\widetilde{H}(A)\oplus\widetilde{H}(A') \xrightarrow{g\oplus g'} \widetilde{H}(X)$ on homology.

But the pinch map

$$S^n \longrightarrow S^n \vee S^n$$



is the map

$$1 \longmapsto (1, 1)$$

on $\tilde{H}_n(S^n)$.

So by functoriality,

$$S^n \longrightarrow S^n \vee S^n \xrightarrow{g \vee g'} X$$

induces the map

$$1 \longmapsto (1, 1) \longmapsto g(1) + g'(1)$$

on $\tilde{H}_n$.

i.e., $$g' \cdot g\,(1) = g'(1) + g(1).$$

//

<u>Thm</u> (Hurewicz)

Let X be a ~~CW complex~~ space, and fix $n \geq 1$.

If $\pi_k(X, x_0) = 0 \ \forall\ k \leq n$, ~~$n \geq 0$~~

then the Hurewicz map

$$\pi_k(X, x_0) \longrightarrow \widetilde{H}_k(X)$$

is an $\cong$ $\forall\ k \leq n+1$.

	If	Then
k	π_k	H_k
0	0 $\Rightarrow$	0
1	0 $\Rightarrow$	0
$\vdots$	$\vdots$	$\vdots$
n	0 $\Rightarrow$	0
$n+1$	A $\cong$	A

<u>Rmk.</u> So $\widetilde{H}_k(X) = 0 \ \forall\ k \leq n$,

and $\widetilde{H}_{n+1}(X) \cong \pi_{n+1}(X)$.

~~Cor The theorem holds for any pointed space X. (By CW approx.)~~

<u>Cor</u> Let $A \hookrightarrow X$ be a cofibration, fix $n \geq 1$,

and let A be 1-connected. Then

if $\pi_k(X, A) = 0 \ \forall\ k \leq n$, the map

$$\pi_k(X, A) \longrightarrow H_k(X, A)$$

is a $\cong$ $\forall\ k \leq n+1$.

Still need to show this is a map of grps.

__Pf of Cor__ Just need to show commutativity of

$$\begin{tikzcd} \pi_k(X,A) \arrow[r, "\text{Hurewicz}"] \arrow[d, "\wr"'] & H_k(X,A) \arrow[d, "\wr"] \\ \pi_k(X/A) \arrow[r, "\text{Hurewicz}"'] & \tilde{H}_k(X/A) \end{tikzcd}$$

$\cong$ for $k \le n+1$ by homotopy excision homework

$\cong$ by first few weeks of class

$\cong$ by Hurewicz theorem.

__Cor__ Let (X, x_0) and (Y, y_0) be CW complexes w/ $\pi_1(X, x_0) \cong \pi_1(Y, y_0) \cong 0$. If $f: (X, x_0) \to (Y, y_0)$ induces an $\cong$ on homology, then f is a homotopy equivalence.

__Pf__ (Homework)

What if $\pi_1 \neq 0$?

<u>Thm</u> (Hurewicz).

Let X be connected. The Hurewicz map induces an isomorphism

$$\underbrace{\pi_1(X, x_0) / [\pi_1 X, \pi_1 X]}_{\text{abelianization of } \pi_1(X, x_0).} \xrightarrow{\simeq} H_1(X).$$

<u>Ex</u> If $\pi_1(X)$ is abelian, $\pi_1(X) \cong H_1(X)$.

<u>Pf</u> Either next class, or homework.

Proof of main theorem requires ~~two lemmas~~ a two-part lemma:

<u>Lemma</u> Fix $m \geq 2$. Then

(1) $\pi_n \left(\bigvee_\alpha S^m \right) \cong \bigoplus_\alpha \mathbb{Z}$.

More generally, fix gluing maps

$$\Phi_\beta : D_\beta^{n+1} \longrightarrow \bigvee_\alpha S^m.$$

Then

(2) $\pi_m \left(\left(\coprod_\beta D^{m+1} \right) \amalg \bigvee_\alpha S^m \Big/ \Phi_\beta \right) \cong \operatorname{coker} \left(\bigoplus_\beta \mathbb{Z} \xrightarrow{\oplus \deg \Phi_\beta} \bigoplus_\alpha \mathbb{Z} \right)$.

<u>Pf</u> (of main theorem, based on ~~corollary~~ lemma)

By CW approximation, $\exists$ CW complex $\tilde{X}$ and a map

$$\tilde{X} \longrightarrow X$$

inducing $\cong$ on π_k $\forall$ k. Further, $\pi_k(X, x_0) = 0$ $\forall$ $k \leq n$ implies we can chose $\tilde{X}$ to have a single 0-cell and no k-cells for $1 \leq k \leq n$. By cellular approximation, $\pi_k(\tilde{X}, x_0) \cong \pi_k(\tilde{X}^{n+2}, x_0)$ $\forall$ $k \leq n+1$. For $k = n+1 = m$, the ~~inclusion~~ lemma tells us

$$\pi_{n+1}(\tilde{X}^{n+2}, x_0) \cong \operatorname{coker} \left(\bigoplus_\beta \mathbb{Z} \xrightarrow{\oplus \deg \Phi_\beta} \bigoplus_\alpha \mathbb{Z} \right)$$

but the RHS is also $H_{n+1}(\tilde{X}^{n+2})$ by the cellular boundary formula. (Note $\bigoplus_\alpha \mathbb{Z} = \ker(\partial_{n+1})$ since cellular chain complex $\cong 0$ in degree n.)

A quick check shows that the identification of (2) in lemma is compatible w/ Hurewicz map, so

$$\pi_{n+1}(X, x_0) \cong H_{n+1}(X)$$

via the Hurewicz map. //

__Pf__ (1st lemma)

(1) Let $Y = \prod_\alpha S^n_\alpha$, and let $B \subset Y$ be the subspace s.t. $b \in B \iff b_\alpha \neq * \in S^n_\alpha$ for ~~only one~~ at most one α.

(If $\{\alpha\}$ is finite, $B \underset{\text{homeo}}{\cong} \bigvee_\alpha S^n_\alpha$.)

~~For $\{\alpha\}$ finite,~~ We have

$$\pi_{n+1}(Y, A) \to \pi_n(B) \to \pi_n(Y) \to \pi_n(Y, A) \to$$

but the pair (Y, A) can be given a CW structure w/ __no__ cells in dim $< 2n$, so $\pi_n(Y, A) \cong \pi_{n+1}(Y, A) \cong 0$. $\to$ $\pi_n B \cong \pi_n Y$.

So if $\{\alpha\}$ is finite, we have

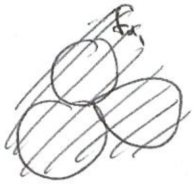
$$\pi_n\left(\bigvee_\alpha S^n_\alpha\right) \cong \pi_n(B) \cong \pi_n(Y) \cong \prod_\alpha \pi_n(S^n) \cong \bigoplus_\alpha \pi_n(S^n)$$

~~$\forall \alpha$, the map $S^n_\alpha \to Y \xrightarrow{\text{projection}} S^n_\alpha$, $x \mapsto (*, \dots, *, x, *, \dots, *) \mapsto x$ factors continuously through B: $S^n_\alpha \hookrightarrow B \to Y \to S^n_\alpha$~~

Moreover, the $\cong$

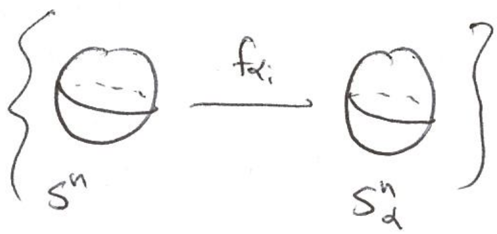
$$\prod_\alpha \pi_n(S^n_\alpha) \overset{h}{\cong} \pi_n\left(\vee_\alpha S^n_\alpha\right)$$

is given by sending

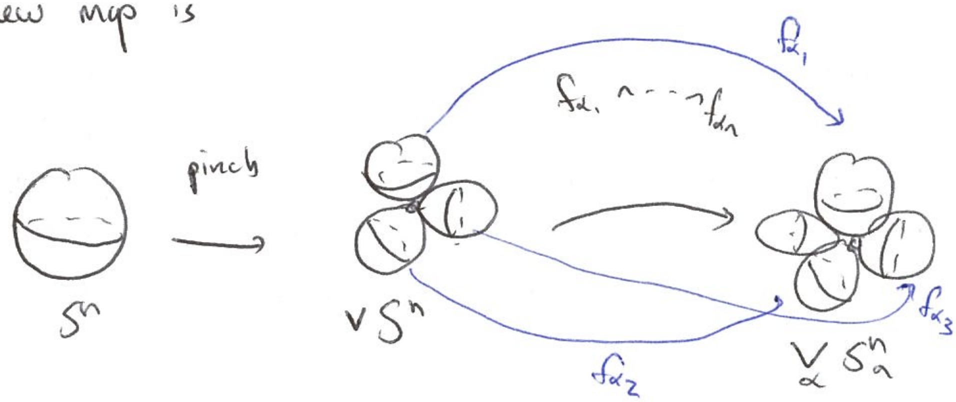


$$(f_{\alpha_1}, \dots, f_{\alpha_n}) \overset{h}{\longmapsto} (f_{\alpha_1} \vee \dots \vee f_{\alpha_n}) \circ \text{pinch}.$$

i.e., given maps



the new map is



Now consider $\bigoplus_\alpha \pi_n(S^n_\alpha) \xrightarrow{h} \pi_n\left(\vee_\alpha S^n_\alpha\right)$ for $\{\alpha\}$ not finite. (Defined same way as above.) This is a surjection — since S^n is cmpct, any $[g] \in \pi_n(\vee_\alpha S^n_\alpha)$ factors thrgh finitely many wedges of S^n.

The map is also an injection: If

$$\tilde{f} : D^{n+1} \longrightarrow \bigvee_{\alpha} S^n_{\alpha}$$

is a null-homotopy of f,

$\tilde{f}$ factors through some finite collection $\{\alpha_j\}$ by compactness of D^{n+1}. Hence

$$[f] = 0 \text{ in } \pi_n\Big(\bigvee_{\alpha} S^n_{\alpha}\Big)$$

$$\Downarrow$$

$$[f] = 0 \text{ in } \pi_n\Big(\bigvee_{\substack{\alpha_j \\ \text{finite}}} S^n_{\alpha_j}\Big) \cong \bigoplus_{\substack{\text{finite} \\ \alpha_j}} \pi_n(S^n_{\alpha_j}).$$

But $\bigoplus \pi_n(S^n_{\alpha_j}) \hookrightarrow \bigoplus \pi_n(S^n_{\alpha})$ is injective, ~~so by commutativity of previous diagram, showing $\pi_n(\bigvee S^n$~~ and $h^{-1}(f) \in \bigoplus \pi_n(S^n_{\alpha_j}) \subset \bigoplus \pi_n(S^n_{\alpha})$

so $h^{-1}(f)$ must be 0 as well.

Let $Y = \left(\coprod D^{m+1}_\beta\right) \bigcup_{\Phi_\beta} \left(\bigvee_\alpha S^m_\alpha\right)$.

Then

$$\pi_{m+1}(Y) \to \pi_{m+1}(Y, \vee_\alpha S^m_\alpha) \xrightarrow{\partial}$$

$$\pi_m(\vee_\alpha S^m_\alpha) \to \pi_m(Y) \to \pi_m(Y, \vee_\alpha S^m_\alpha).$$

$$\begin{array}{c} \| \\ 0 \end{array}$$ since Y obtained by attaching $(m+1)$-cells

Further,

$$\pi_{m+1}(Y, \vee_\alpha S^m_\alpha) \to \pi_{m+1}(Y/\vee S^m_\alpha)$$

$$\cong \pi_{m+1}(\vee_\beta S^{m+1}_\beta)$$

is $\cong$ since $(Y, \vee_\alpha S^m_\alpha)$ is m-connected, and $\vee_\alpha S^m_\alpha$ is $(m-1)$-conn.

So $\pi_m(Y) \cong \pi_m(\vee_\alpha S^m_\alpha) / \operatorname{image}(\partial)$ by tracing through $\cong$ between $\pi_{m+1}(Y, \vee S^m)$ and $\pi_{m+1}(\vee_\beta S^{m+1})$

$$\cong \operatorname{Coker}\left(\bigoplus \mathbb{Z}_\beta \xrightarrow{\Sigma \deg \Phi_\beta} \bigoplus \mathbb{Z}_\alpha\right)$$

//.