Künneth Formula + Universal Coefficient Theorem

Last time we started talking about homology w/ coefficients.

Given π abelian grp,

$$C_*(X;\pi) := C_*(X) \otimes \pi$$

$$H_*(X;\pi) := H_*(C_*(X;\pi)).$$

We claimed

Thm (0) $H_*(pt;\pi) \cong \begin{cases} \pi & *=0 \\ 0 & \text{other} \end{cases}$

(1)

(2) Excision

(3) $H_*(\amalg X_i, \amalg A_i) \cong \bigoplus H_*(X_i, A_i)$

(4) $f: X \simeq Y$ w.h.e. $\Rightarrow f_*: H_* X \to H_* Y$ isomorphism

$\Uparrow$

(4') $f, g: (X,A) \to (Y,B)$ and f, g homotopic as a map of pairs $\Longrightarrow f_* = g_*$ on homology.

Cor $H_k(S^n;\pi) \cong \begin{cases} \pi & k=n,0 \\ 0 & \text{other} \end{cases}$

(by induction on Mayer-Vietoris)

Cor Let X CW complex $C^{CW}_*(X;\pi) := C^{CW}_*(X) \otimes \pi$.

Then $H_*(X;\pi) \cong H_*(C^{CW}_*(X;\pi))$.

How do you compute H_* w/ coeff in π? Same ways as before (Mayer-Vietoris, etc) but also:

Thm (Künneth, baby version) Let $A_\cdot$ be chain complex s.t. A_n free $\forall n$.

$\forall n, \exists$ S.E.S.

$$0 \longrightarrow H_n(A) \otimes \pi \longrightarrow H_n(A;\pi) \longrightarrow \underbrace{\mathrm{Tor}_1(H_{n-1}(A);\pi)}_{\text{will define now.}} \longrightarrow 0$$

Natural in A, π.

Splits (NOT naturally.)

Defn Let A, π be two abelian groups.
Let $F_\cdot$ be a chain complex s.t. each F_n is a free abelian group, and $H_k(F_\cdot) = \begin{cases} A & k=0 \\ 0 & \text{other} \end{cases}$

Then
$$\mathrm{Tor}_1(A,\pi) := H_1(F_\cdot \otimes \pi).$$

Ex. $A = \pi = \mathbb{Z}/2\mathbb{Z}$.

Choose $F_\cdot = \underset{}{0} \leftarrow \underset{0}{\mathbb{Z}} \xleftarrow{\times 2} \underset{1}{\mathbb{Z}} \leftarrow 0 \leftarrow 0 \cdots$

Then $F_\cdot \otimes \pi = 0 \leftarrow \underset{0}{\mathbb{Z}/2\mathbb{Z}} \xleftarrow{0} \underset{1}{\mathbb{Z}/2\mathbb{Z}} \leftarrow 0 \leftarrow$

So $H_0(F_\cdot \otimes \pi) = H_1(F_\cdot \otimes \pi) = \mathbb{Z}/2\mathbb{Z}$.

i.e., $\mathrm{Tor}_1(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$.

Exer Compute

- $\mathrm{Tor}_1(\mathbb{Z}, \mathbb{Z})$
- $\mathrm{Tor}_1(\mathbb{Z}, \mathbb{Z}/2\mathbb{Z})$
- $\mathrm{Tor}_1(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z})$.

Facts
- $\mathrm{Tor}_1(A,B) = \mathrm{Tor}_1(B,A)$
- Tor is independent of choice of $F_\cdot$ (so long as $H_0(F_\cdot) = A$, $H_k(F_\cdot) = 0 \ \forall\, k \neq 0$.)

Ex What is $H_2(\mathbb{RP}^2; \mathbb{Z}/2\mathbb{Z})$?

By Künneth,
$$0 \to H_2(\mathbb{RP}^2) \otimes \mathbb{Z}/2\mathbb{Z} \to H_2(\mathbb{RP}^2; \mathbb{Z}/2\mathbb{Z}) \to \mathrm{Tor}_1(H_1(\mathbb{RP}^2), \mathbb{Z}/2\mathbb{Z}) \to 0$$
is S.E.S.

Well, $H_2(\mathbb{RP}^2) = 0$, and $H_1(\mathbb{RP}^2) \cong \mathbb{Z}/2\mathbb{Z}$. So
$$0 \to 0 \to H_2(\mathbb{RP}^2; \mathbb{Z}/2\mathbb{Z}) \to \mathrm{Tor}_1(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/2\mathbb{Z}) \to 0$$
is S.E.S. By above, $H_2(\mathbb{RP}^2; \mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$.

By cellular homology,
$$C_*^{cw}(\mathbb{RP}^2; \mathbb{Z}/2\mathbb{Z}) : 0 \leftarrow \underset{0}{\mathbb{Z}/2\mathbb{Z}} \xleftarrow{\times 0} \underset{1}{\mathbb{Z}/2\mathbb{Z}} \xleftarrow{\times 0} \underset{2}{\mathbb{Z}/2\mathbb{Z}} \leftarrow 0 \leftarrow 0 \cdots \checkmark$$

Homology of product spaces

Given two CW complexes X, Y recall $X \times Y$ $(= k(X \overset{\text{usual}}{\times} Y))$ has CW structure as follows:

- $X \times Y$ has a k cell for every pair $(\alpha, \beta) \in A^X_p \times A^Y_q$ where $p+q=n$.

set of p-cells in X

set of q-cells in Y

- The attaching map

$$\Phi_{\alpha,\beta} : \partial D^k \longrightarrow (X \times Y)^{k-1}$$
$$\partial D^k = (\partial D^p \times D^q) \cup (D^p \times \partial D^q)$$

is given by

$$\left(\Phi_\alpha \times (D^q \hookrightarrow Y^q)\right) \cup \left((D^p \hookrightarrow X^p) \times \Phi_\beta\right).$$

Where by induction we see $(X^{p-1} \times Y^q) \cup (X^p \times Y^{q-1})$ is a sub-CW complex of $(X \times Y)^{k-1}$.

$\underline{\underline{Ex}}$ $X = Y = S^1$.

Choose CW structure so

$$A_i^X = \{\alpha_i\},\ \text{a single } i\text{-cell in dim } i = 0,1.$$

$$A_j^X = \phi \quad \forall\, j \neq 0,1.$$

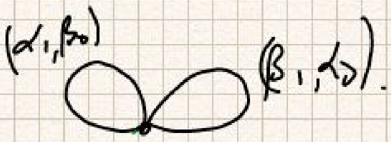
Likewise, $A_i^Y = \begin{cases} \{\beta_i\} & i = 0,1 \\ \phi & \text{otherwise.} \end{cases}$

Then

$$A_0^{X\times Y} = \{(\alpha_0, \beta_0)\}$$

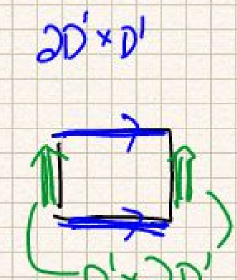
$$A_1^{X\times Y} = \{(\alpha_1, \beta_0),\ (\alpha_0, \beta_1)\}$$

$$A_2^{X\times Y} = \{(\alpha_1, \beta_1)\}.$$

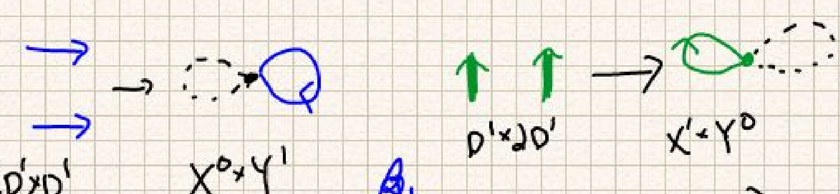
$(X\times Y)^1 =$ 

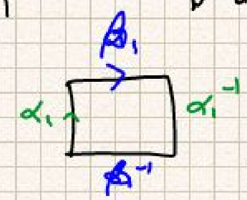
The attaching map $\Phi_{\alpha_1,\beta_1} : \partial D^2 \longrightarrow (X\times Y)^1$

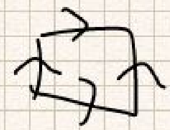
$$\partial D^2 = (\partial D^1 \times D^1) \cup (D^1 \times \partial D^1) =$$



is given by $\left(\Phi_{\alpha_1} \times (D^1 \hookrightarrow Y^1)\right) \cup \left((D^1 \hookrightarrow X^1) \times \Phi_{\beta_1}\right)$.



So gluing map is by 

. or, .

Can we express cellular chain complex of $X \times Y$ in terms of $C_*^{cw}(X)$ and $C_*^{cw}(Y)$?

Prop.

$$C_*^{cw}(X \times Y) \cong C_*^{cw}(X) \otimes C_*^{cw}(Y).$$

Sketch of pf. By definition of $\otimes$ of chain complexes,

$$(A \otimes B)_n := \bigoplus_{p+q=n} A_p \otimes B_q$$

so $C_n^{cw}(X \times Y) \cong \left(C_\cdot^{cw}(X) \otimes C_\cdot^{cw}(Y)\right)_n$.

One can check that

$$\partial_n^{cw}(\alpha_p \otimes \beta_q) = \partial_p^{cw}\alpha_p \otimes \beta_q + (-1)^p \alpha_p \otimes \partial_q^{cw} \beta_q.$$

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