So can we compute
$H_\bullet(X \times Y)$?

<u>Thm</u> (Künneth)

Let $A_\bullet, B_\bullet$ be chain complexes.
Assume A_n is free ab. grp. $\forall n$. Then $\forall n$,
$\exists$ S.E.S.

$$0 \to \bigoplus_{p+q=n} H_p(A) \otimes H_q(B) \to H_n(A \otimes B) \to \bigoplus_{p+q=n-1} \mathrm{Tor}_1(H_pA, H_qB) \to 0.$$

This is natural in A and B.

The sequence splits, but the splitting is not natural.

Since $C_\bullet^{CW}(X \times Y) \cong C_\bullet^{CW}(X) \otimes C_\bullet^{CW}(Y)$,

plugging in $A = C_\bullet^{CW}(X)$, $B = C_\bullet^{CW}(Y)$

does the trick.

<u>Ex</u> $X = Y = \mathbb{R}P^2$. Then

$$0 \to \bigoplus_{p+q=4} H_pX \otimes H_qY \to H_4(X \times Y) \to \bigoplus_{p+q=3} \mathrm{Tor}_1(H_pX, H_qY) \to 0.$$

$H_pX = 0 \ \forall p \le 2$,
so $p+q = 2+2 = 4$.
But $H_2X = H_2Y = 0$.

(p,q) must equal $(2,1)$ or $(1,2)$.
But $H_2(X) = 0$, so
$\mathrm{Tor}(H_2X, H_1Y) = 0$.

simplifies to the SES

$$0 \to 0 \hookrightarrow H_4(X \times Y) \twoheadrightarrow 0 \to 0.$$

$\Rightarrow H_4(\mathbb{R}P^2 \times \mathbb{R}P^2) \cong 0$.

$n=3$:

$$0 \to \bigoplus_{p+q=3} H_p(X) \otimes H_q(Y) \to H_3(X \times Y) \to \bigoplus_{p+q=2} \mathrm{Tor}_1(H_pX, H_qY) \to 0$$

$$\bigoplus_{p+q=3} H_p(X) \otimes H_q(Y) \cong H_1(X) \otimes H_2(Y) \oplus H_2(X) \otimes H_1(Y) \cong 0$$

$$\bigoplus_{p+q=2} \mathrm{Tor}_1(H_pX, H_qY) \cong \mathrm{Tor}(H_1X, H_1Y) \oplus \mathrm{Tor}(H_2X, H_0Y) \oplus \mathrm{Tor}(H_0X, H_2Y) \cong \mathbb{Z}/2\mathbb{Z} \oplus 0 \oplus 0$$

$\Rightarrow H_3(\mathbb{R}P^2 \times \mathbb{R}P^2) \cong \mathbb{Z}/2\mathbb{Z}$.

$n=2$: $H_2(\mathbb{R}P^2 \times \mathbb{R}P^2) \cong \mathbb{Z}/2\mathbb{Z}$ $\qquad H_0 = \mathbb{Z}$.

$H_1(\quad) \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$. $\qquad$ (Connected.)

<u>Cor.</u> If $H_p(X)$ is free $\forall p$, then

$$H_n(X \times Y) \cong \bigoplus_{p+q=n} H_p(X) \otimes H_q(Y)$$

$\forall$ spaces Y.

The following may be useful in the future:

<u>Thm</u> Let R be a PID (ex $R = \mathbb{Z}$, $R = k[x]$, $R =$ field)

Let A, B be chain complexes of R-modules, s.t. A_n is flat $\forall n$. (ex A_n free).

Then $\exists$ SES

$$0 \to \bigoplus_{p+q=n} H_p(A) \otimes_R H_q(B) \to H_n(A \otimes_R B) \to \underbrace{\mathrm{Tor}_1^R(H_pA, H_qB)}_{\bigoplus_{p+q=n-1}} \to 0$$

natural in $A_\bullet, B_\bullet$.

This splits, though not naturally.

<u>Rmk</u> $\mathrm{Tor}_1^R(M,N) := H_1(F_\bullet \otimes_R N)$, where $F_\bullet$ is a chain complex of free R-modules s.t. $H_k(F_\bullet) = \begin{cases} M & k=0 \\ 0 & \text{o/w.} \end{cases}$

Cohomology.

Given a chain complex $A_\bullet$, we can form a cochain complex

$$\hom^\bullet(A,\pi)$$

for any abelian group π.

This is a sequence of groups

$$\hom(A_0,\pi) \xrightarrow{d^0} \hom(A_1,\pi) \xrightarrow{d^1} \cdots$$

where d^i is defined by

$$(d^i f)(a) = (-1)^i f(\partial a).$$

<u>Defn</u> A cochain complex $A^\bullet$ is a sequence of abelian groups $\{A^i\}_{i\in\mathbb{Z}}$ together with maps

$$d^i : A^i \to A^{i+1}$$

such that $d^2 = d^{i+1}\circ d^i = 0$.

The <u>k^{th} cohomology group of $A^\bullet$</u> is

$$H^k(A^\bullet) := \mathrm{Ker}(d^k)/\mathrm{Image}(d^{k-1}).$$

<u>Rmk</u> A cochain complex $A^\bullet$ is the same thing as a chain complex $a_\bullet$ by setting

$$a_i = A^{-i}.$$

So

$$\cdots \to A^{-1} \xrightarrow{d} A^0 \xrightarrow{d} A^1 \xrightarrow{d} A^2 \to \cdots$$

$$\Updownarrow$$

$$\begin{array}{ccccccc} & A^1 & & A^0 & & A^{-1} & \\ & \| & & \| & & \| & \\ \cdots \leftarrow & a_{-1} & \leftarrow & a_0 & \leftarrow & a_1 & \leftarrow \cdots \end{array}$$

(So this isn't some new, scary thing.)

<u>Defn</u> Let X be a topological space, and π an abelian group.

The <u>singular cochain complex of X with coefficients in π</u> is the cochain complex

$$\hom^\bullet(C_\bullet(X),\pi) =: C^\bullet(X;\pi)$$

The k^{th} cohomology group of X w/ coeffs in π is

$$H^k(X;\pi) := H^k(\hom^\bullet(C_\bullet(X),\pi)).$$

$$C^\bullet(X) := C^\bullet(X;\mathbb{Z})$$
$$H^\bullet(X) := H^\bullet(X;\mathbb{Z}).$$

$H_\bullet$ was computable because of long exact sequences. We quickly develop the same technology for $H^\bullet$.

<u>Propn</u> $C^\bullet(-,\pi)$ defines a functor

$$\mathrm{Spaces}^{op} \to \mathrm{Cochain}.$$

<u>Pf.</u> Any map of chain cmplxs

$$f : A_\bullet \to B_\bullet$$

induces a map

$$\hom^\bullet(B_\bullet,\pi) \to \hom^\bullet(A_\bullet,\pi)$$
$$\phi \longmapsto \phi\circ f.$$

Any any continuos map

$$f : X \to Y$$

gives a map of chain cmplxs

$$C_\bullet(X) \to C_\bullet(Y).$$

You can check the rest. //

What does $H^\cdot$ measure?

Let $\psi \in C^k(X)$ and $d\psi = 0$, so $[\psi] \in H^k(X)$.

$d\psi = 0 \iff \psi(\partial\sigma) = 0 \;\forall\, \sigma \in C_{k+1}$

$\Rightarrow \psi(\tau) = \psi(\tau')$ if $\tau - \tau' = \partial\sigma$.

So an element of H^* can only tell apart chains if they don't ~~bound together~~ bound something.

(Compare: $H_\cdot$ identifies chains if they together bound something.)

In fact, any $[\psi] \in H^k(X)$ defines a map

$$H_k(X) \longrightarrow \mathbb{Z}$$

and every map arises from H^k, as we will see.

Now, $[\psi] = 0 \iff \psi = d\phi$.

But $\psi = d\phi \Rightarrow \psi(\tau) = d\phi(\tau) = \pm\phi(\partial\tau)$.

So $[\psi] = 0$ if its values on a chain depend only on the boundary of the chain.

(For example, $[\psi] = 0 \Rightarrow \psi(\text{any closed chain}) = 0$.)

What properties does cohomology satisfy?

Rmk $H_\cdot(X)$ does NOT form a graded "coalgebra." $C_\cdot(X)$ does in a suitable sense. One reason $H^\cdot$ is useful is that we obtain a useful ring structure after passing to $H^\cdot$ (which is easier than $C_\cdot$ or $C^\cdot$ itself).

Thm.

(Dimension)
$$H^k(pt) \cong \begin{cases} \mathbb{Z} & k=0 \\ 0 & \text{else} \end{cases}$$

(Exactness) $\forall$ pairs (X,A), we have LES

$$\cdots \to H^k(X,A) \longrightarrow H^k(X) \to H^k(A) \to H^{k+1}(X,A) \to \cdots$$

(Excision) $\forall\, Z \subset A \subset X$ s.t. $\overline{Z} \subset \operatorname{int}(A)$, the map

$$H^k(X \setminus Z, A \setminus Z) \leftarrow H^k(X,A)$$

is an $\cong$ $\forall k$

(Products)
$$H^k\Big(\coprod_i X_i, \coprod A_i\Big) \cong \prod_i H^k(X_i, A_i)$$

(Hmtpy)
If $f, g : (X,A) \to (Y,B)$ are homotopic, then $f^* = g^*$ on H^*.

We'll define $H^*(X,A)$ in a second. But there's one property that really distinguishes H^* from H_*.

Thm $\forall$ spaces X, $\exists$ maps

$$H^i(X) \otimes H^j(X) \longrightarrow H^{i+j}(X)$$

that turn $H^\bullet(X) := \bigoplus_k H^k(X)$ into a (graded) ring, w/ unit

$\forall\, f : X \to Y$,
$$f^* : H^\bullet(Y) \to H^\cdot(X)$$
is a map of graded rings.

Cor H^* is a functor

$$\text{Spaces}^{op} \longrightarrow \text{Gr Rings}$$

We'll study ring structure later.

If $A \hookrightarrow X$ is an inclusion,

$$C_n(A) \to C_n(X)$$

is an injection $\forall n$, and

$$C^n(X) \to C^n(A)$$

is a surjection.

<u>Defn</u> The n^{th} relative cochain group of (X,A) is

$$C^n(X,A) := \mathrm{Ker}(C^n(X) \to C^n(A)).$$

The cohomology of the cochain complex $C^\cdot(X,A)$ is the <u>relative</u> cohomology of the pair (X,A), denoted

$$H^\cdot(X,A).$$

<u>Rmk</u> Since $C_\cdot(A) \hookrightarrow C_\cdot(X)$ is an inclusion of free groups,

$$C^\cdot(X,A;G) \cong \hom^\cdot(C_\cdot(X,A);G).$$

$C^n(X,A)$ consists of functions

$$\phi: C_n(X) \to \mathbb{Z}$$

which vanish on simplices contained entirely in $A \subset X$.

<u>Propn</u> Any pair (X,A) gives rise to a LES of cohomology groups

$$0 \to H^0(X,A) \to H^0(X) \to H^0(A) \to$$
$$\to H^1(X,A) \to H^1(X) \to H^1(A) \to$$
$$\to H^2(X,A) \to \cdots$$

<u>Pf</u> $\forall n$, we have a SES

$$0 \leftarrow C^n(A) \leftarrow C^n(X) \leftarrow C^n(X,A) \leftarrow 0.$$

So the same proof as in week 2 goes through. //

Before, we saw how to compute $H_\cdot(A \otimes B)$.

Can we compute $H^\bullet(\hom(A_\cdot;\pi))$ knowing $H_\cdot(A)$?

<u>Defn</u> Given abelian groups A and π, let

$$\begin{array}{ccccccc} F_2 \to & F_1 \to & F_0 & \to 0 \\ & & \downarrow & \\ 0 \to & 0 \to & A & \to 0 \end{array}$$

be a free resolution of A.

Then

$$H^k(\hom^\bullet(F_\cdot,\pi)) := \mathrm{Ext}^k(A,\pi).$$

<u>Rmk</u>:
- You should think of Ext as a fancy version of Hom
- Ext is indep. of free resolution.
- Unlike Tor, $\mathrm{Ext}^k(A,B) \neq \mathrm{Ext}^k(B,A)$ just as $\hom(A,B) \neq \hom(B,A)$.

Thm. Let $A_\cdot$ be a chain complex s.t. A_n is free $\forall n$. Then $\exists$ a SES

Universal Coefficient Theorem!

$$0 \to \mathrm{Ext}^1(H_{n-1}(A),\pi) \to H^n(\hom^\cdot(A,\pi)) \to \hom(H^n(A),\pi) \to 0.$$

This sequence is natural in $A_\cdot$ and π.

This sequence is split, but the splitting is NOT natural.

Exer Compute

- $\mathrm{Ext}^1(\mathbb{Z}/n\mathbb{Z},\mathbb{Z})$
- $\mathrm{Ext}^1(\mathbb{Z},\pi)$
- $\mathrm{Ext}^1(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}/n\mathbb{Z})$.

Ans:

- $0 \to \hom(\mathbb{Z},\mathbb{Z}) \xrightarrow{\times n} \hom(\mathbb{Z},\mathbb{Z}) \to 0$

  $\mathrm{Ext}^1(\mathbb{Z}/n\mathbb{Z},\mathbb{Z}) \cong \mathbb{Z}/n\mathbb{Z}$
- $0 \to \hom(\mathbb{Z},\pi) \to \hom(0,\pi) \to 0 \cdots$

  $\mathrm{Ext}^1(\mathbb{Z},\pi) \cong 0.$
- $0 \to \hom(\mathbb{Z},\mathbb{Z}/n\mathbb{Z}) \xrightarrow{\times n} \hom(\mathbb{Z},\mathbb{Z}/n\mathbb{Z}) \to 0$

  $\mathrm{Ext}^1(\mathbb{Z}/n\mathbb{Z},\mathbb{Z}/n\mathbb{Z}) \cong \mathbb{Z}/n\mathbb{Z}$.

Rmk The surjection

$$H^n(\hom^\cdot(A_\cdot,\pi)) \xrightarrow{j} \hom(H_n(A),\pi)$$

is not hard to see. First, note that

$$d\psi = 0 \iff \psi(\partial a) = 0 \quad \forall a \in A_{n-1}$$

Define

$$j_*[\psi] : H_n(A) \to \pi$$

by

$$(j_*[\psi])([a]) = \psi(a).$$

Well,
$$\begin{aligned}(\psi + d\phi)(a+\partial b) &= \psi(a) + d\phi(a) + \psi(\partial b) + d\phi(\partial b)\\ &= \psi(a) \pm \phi(\partial a) \pm d\psi(b) \pm \phi(\partial^2 b)\\ &= \psi(a) \pm 0 \pm 0 \pm 0.\end{aligned}$$

since $\partial a = 0$ and $d\psi = 0$

Hence j_* is well-defined. Surjection via hmwk.

Exer Compute $H^\cdot(\mathbb{RP}^2)$.

H^0: $0 \to \underbrace{\mathrm{Ext}^1(H_{-1}(\mathbb{RP}^2),\mathbb{Z})}_{=0} \to H^0(\mathbb{RP}^2) \to \hom(H_0\mathbb{RP}^2,\mathbb{Z}) \to 0$

$\Rightarrow H^0(\mathbb{RP}^2) \cong \mathbb{Z}$.

H^1: $0 \to \underbrace{\mathrm{Ext}^1(H_0\mathbb{RP}^2,\mathbb{Z})}_{=0} \to H^1(\mathbb{RP}^2) \to \underbrace{\hom(\underbrace{H_1\mathbb{RP}^2}_{\mathbb{Z}/2\mathbb{Z}},\mathbb{Z})}_{=0} \to 0$

$\Rightarrow H^1(\mathbb{RP}^2) \cong \mathbb{Z}$.

H^2: $0 \to \underbrace{\mathrm{Ext}^1(\underbrace{H_1\mathbb{RP}^2}_{=\mathbb{Z}/2\mathbb{Z}},\mathbb{Z})}_{\mathbb{Z}/2\mathbb{Z}} \to H^2(\mathbb{RP}^2) \to \hom(\underbrace{H_2\mathbb{RP}^2}_{\substack{\pi\\0}},\mathbb{Z}) \to 0$

$\Rightarrow H^2(\mathbb{RP}^2) \cong \mathbb{Z}/2\mathbb{Z}$.