If $A \hookrightarrow X$ is an inclusion,

$$C_n(A) \to C_n(X)$$

is an injection $\forall n$, and

$$C^n(X) \to C^n(A)$$

is a surjection.

Defn The n^{th} relative cochain group of (X,A) is

$$C^n(X,A) := \operatorname{Ker}(C^n(X) \to C^n(A)).$$

The cohomology of the cochain complex $C^\bullet(X,A)$ is the <u>relative</u> cohomology of the pair (X,A), denoted

$$H^\bullet(X,A).$$

Rmk Since $C_\bullet(A) \hookrightarrow C_\bullet(X)$ is an inclusion of free groups,

$$C^\bullet(X,A;G) \cong \hom^\bullet(C_\bullet(X,A);G).$$

$C^n(X,A)$ consists of functions

$$\phi: C_n(X) \to \mathbb{Z}$$

which vanish on simplices contained entirely in $A \subset X$.

Propn Any pair (X,A) gives rise to a LES of cohomology groups

$$0 \to H^0(X,A) \to H^0(X) \to H^0(A) \to$$
$$\to H^1(X,A) \to H^1(X) \to H^1(A) \to$$
$$\to H^2(X,A) \to \cdots$$

Pf $\forall n$, we have a SES

$$0 \leftarrow C^n(A) \leftarrow C^n(X) \leftarrow C^n(X,A) \leftarrow 0.$$

So the same proof as in week 2 goes through. //

Before, we saw how to compute $H_\bullet(A \otimes B)$.

Can we compute $H^\bullet(\hom(A_\bullet;\pi))$ knowing $H_\bullet(A)$?

Defn Given abelian groups A and π, let

$$\begin{array}{ccccccc} F_2 & \to & F_1 & \to & F_0 & \to & 0 \\ & & & & \downarrow & & \\ 0 & \to & 0 & \to & A & \to & 0 \end{array}$$

be a free resolution of A. Then

$$H^k(\hom^\bullet(F_\bullet,\pi)) := \operatorname{Ext}^k(A,\pi).$$

Rmk:
- You should think of Ext as a fancy version of Hom
- Ext is indep. of free resolution.
- Unlike Tor, $\operatorname{Ext}^k(A,B) \neq \operatorname{Ext}^k(B,A)$ just as $\hom(A,B) \neq \hom(B,A)$.

Thm. Let $A_\cdot$ be a chain complex s.t. A_n is free $\forall n$. Then $\exists$ a SES

$\boxed{\text{Universal Coefficient Theorem!}}$

$$0 \to \mathrm{Ext}^1(H_{n-1}(A), \pi) \to H^n(\mathrm{hom}^\cdot(A, \pi)) \to \mathrm{hom}(H^n(A), \pi) \to 0.$$

This sequence is natural in $A_\cdot$ and π.

This sequence is split, but the splitting is NOT natural.

Exer Compute

- $\mathrm{Ext}^1(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z})$
- $\mathrm{Ext}^1(\mathbb{Z}, \pi)$
- $\mathrm{Ext}^1(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}/n\mathbb{Z})$.

Ans:

- $0 \to \mathrm{hom}(\mathbb{Z}, \mathbb{Z}) \xrightarrow{\times n} \mathrm{hom}(\mathbb{Z}, \mathbb{Z}) \to 0$

  $\mathrm{Ext}^1(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}) \cong \mathbb{Z}/n\mathbb{Z}$
- $0 \to \mathrm{hom}(\mathbb{Z}, \pi) \to \mathrm{hom}(0, \pi) \to 0 \cdots$

  $\mathrm{Ext}^1(\mathbb{Z}, \pi) \cong 0.$
- $0 \to \mathrm{hom}(\mathbb{Z}, \mathbb{Z}/n\mathbb{Z}) \xrightarrow{\times n} \mathrm{hom}(\mathbb{Z}, \mathbb{Z}/n\mathbb{Z}) \to 0$

  $\mathrm{Ext}^1(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}/n\mathbb{Z}) \cong \mathbb{Z}/n\mathbb{Z}.$

Rmk The surjection

$$H^n(\mathrm{hom}^\cdot(A_\cdot, \pi)) \xrightarrow{j} \mathrm{hom}(H_n(A), \pi)$$

is not hard to see. First, note that

$$d\psi = 0 \iff \psi(\partial a) = 0 \quad \forall a \in A_{n-1}$$

Define

$$j_*[\psi] : H_n(A) \to \pi$$

by

$$(j_*[\psi])([a]) = \psi(a).$$

Well,
$$\begin{aligned}(\psi + d\phi)(a + \partial b) &= \psi(a) + d\phi(a) + \psi(\partial b) + d\phi(\partial b) \\ &= \psi(a) \pm \phi(\partial a) \pm d\psi(b) \pm \phi(\partial^2 b) \\ &= \psi(a) \pm 0 \pm 0 \pm 0.\end{aligned}$$

since $\partial a = 0$ and $d\psi = 0$

Hence j_* is well-defined. Surjection via hmwk.

Exer Compute $H^\cdot(\mathbb{RP}^2)$.

H^0: $0 \to \underbrace{\mathrm{Ext}^1(H_{-1}(\mathbb{RP}^2), \mathbb{Z})}_{=0} \to H^0(\mathbb{RP}^2) \to \mathrm{hom}(H_0\mathbb{RP}^2, \mathbb{Z}) \to 0$

$$\Rightarrow H^0(\mathbb{RP}^2) \cong \mathbb{Z}.$$

H^1: $0 \to \underbrace{\mathrm{Ext}^1(H_0\mathbb{RP}^2, \mathbb{Z})}_{=0} \to H^1(\mathbb{RP}^2) \to \underbrace{\mathrm{hom}(\underbrace{H_1\mathbb{RP}^2}_{\mathbb{Z}/2\mathbb{Z}}, \mathbb{Z})}_{=0} \to 0$

$$\Rightarrow H^1(\mathbb{RP}^2) \cong \mathbb{Z}.$$

H^2: $0 \to \underbrace{\mathrm{Ext}^1(\underbrace{H_1\mathbb{RP}^2}_{=\mathbb{Z}/2\mathbb{Z}}, \mathbb{Z})}_{\mathbb{Z}/2\mathbb{Z}} \to H^2(\mathbb{RP}^2) \to \mathrm{hom}(\underbrace{H_2\mathbb{RP}^2}_{=0}, \mathbb{Z}) \to 0$

$$\Rightarrow H^2(\mathbb{RP}^2) \cong \mathbb{Z}/2\mathbb{Z}.$$

Graded rings

(1) **Defn** A graded ring H is the data of

- A sequence of abelian groups H^n $\forall n \in \mathbb{Z}$ s.t. $H = \bigoplus_n H^n$ as a group, and
- An associative map $\cup : H \otimes H \to H$

such that $\forall\, k, l \in \mathbb{Z}$, we have

$$\cup\big|_{H^k \otimes H^l} : H^k \otimes H^l \to H^{k+l} \subset H.$$

Rmk $H \otimes H = (\oplus H^k) \otimes (\oplus H^l)$

$$\cong \bigoplus_{k+l=n} H^k \otimes H^l$$

This explains what we mean by restricting $\cup$ to $H^k \otimes H^l \subset H \otimes H$.

(2) **Defn** H is called unital if $\exists\ 1 \in H^0$ s.t.

$$1 \cup \alpha = \alpha \cup 1 = \alpha \quad \forall \alpha \in H.$$

(4)
Ex Let L^n be the group of all homogeneous degree n polynomials in x, w/ coeffs in a fixed ring R, commutative.

$$L^n = \{ r_n x^n \}.$$

Then

$$L = \oplus L^n$$

is the ring of Laurent polynomials over R.

(5) **Ex** Let $P^n = L^n$ $\forall\, n \geq 0$, and $P^n = 0$ $\forall\, n < 0$.
Then

$$P = \oplus P^n$$

is the ring of polynomials over R in one variable.

Defn An element $\alpha \in H^k$ is called a degree k element. The degree of α is written

$$|\alpha|.$$

Rmk Conflict of terminology.
Usually, a "degree n polynomial" is any element

$$\sum_{0 \leq k \leq n} r_k x^k \in \bigoplus_{k \leq n} P^n$$

s.t.

$$r_n \neq 0.$$

We won't run into any issues w/ this.

(3) **Defn** A map $f : H \to K$ of graded (unital) rings is a sequence of maps

$$f^k : H^k \to K^k$$

s.t.

$$f^{k+l}(\alpha \cup \beta) = f^k(\alpha) \cup f^l(\beta)$$

$\forall\, \alpha \in H^k,\ \beta \in H^l$.

$$(\text{s.t. } f^0(1_H) = 1_K.)$$

Ex Let $Q^n = 0 \ \forall n$ odd

$$Q^n = P^{\frac{n}{2}} \ \forall n \text{ even.}$$

Then $Q = \bigoplus Q^n$ is often written as

$$R[x], \quad |x| = 2,$$

i.e., as a polynomial ring w/ a generator in degree 2.

We'll see $H^*(\mathbb{CP}^\infty) \cong \mathbb{Z}[x], |x|=2$.

Defn A graded ring H is called graded commutative if

$$\alpha \cup \beta = (-1)^{|\alpha||\beta|} \beta \cup \alpha$$

$\forall \alpha, \beta$.

Ex Q is graded commutative.

In general, the subring

$$\bigoplus H^{2n} \subset \bigoplus H^n$$

is always commutative on the nose.

P is NOT graded commutative.

Ex If α is in odd degree,

$$\alpha^2 = 0.$$

Rmk If we fix a ring R, we can define notion of a graded R-algebra, $H = \bigoplus H^n$, where H is an R-module w/ mult.

$$H \otimes_R H \to H.$$

If $R = \mathbb{Z}/2\mathbb{Z}$, graded commutativity is commutativity.

Rmk (Koszul sign rule) How do remember all these signs? Whenever you study (co)chain complexes, and you slide two symbols α, β past each other, a sign of $(-1)^{|\alpha||\beta|}$ should pop out.

Ex $\partial(\sigma \otimes \tau) := \partial\sigma \otimes \tau + (-1)^{|\sigma|} \sigma \otimes \partial\tau$.

In second term, we slide σ and ∂ past each other. ∂ is a degree ± 1 operator, and $(-1)^{|\sigma||\partial|} = (-1)^{|\sigma|}$.

Thm Singular cohomology defines a functor

$$\text{Spaces}^{op} \to \text{GCRings}$$
$$X \mapsto H^*(X) = \bigoplus H^n(X).$$

$H^*(X)$ is unital so long as $X \neq \emptyset$.

Also, homotopic maps are sent to the same map of graded rings.

Rmk $\forall R$ commutative ring, cohomology w/ coefficients in R defines a functor to GC R-algebras.