<u>Prop'n</u> Let A, B be chain complexes. The maps

$$(1)\quad H_p(A) \otimes H_q(B) \to H_{p+q}(A \otimes B)$$
$$[a] \otimes [b] \mapsto [a \otimes b]$$

and

$$(2)\quad H^q(\hom^\bullet(A, \mathbb{Z})) \xrightarrow{h} \hom(H_q(A), \mathbb{Z})$$
$$[\phi] \longmapsto [a] \mapsto \phi(a)$$

are well-defined.

If A_n is free $\forall\, n$,

(1) is an injection, and

(2) is a surjection.

<u>Pf</u> Well-defined?

(1) $\partial a = 0, \partial b = 0$

$$\Rightarrow \partial(a \otimes b) = \partial a \otimes b + (-1)^{|a|} a \otimes \partial b$$
$$= 0 + 0.$$

If $a' = a + \partial c$
$b' = b + \partial d,$

then

$$(a' \otimes b') = a \otimes b + a \otimes \partial d + \partial c \otimes b + \partial c \otimes \partial d.$$

But $a \otimes \partial d = (-1)^q \partial(a \otimes d)$
$\partial c \otimes b = \partial(c \otimes b)$
$\partial c \otimes \partial d = \partial(c \otimes \partial d)$

since a, b are closed.

Hence

$$a' \otimes b' = a \otimes b + \partial(\text{something}).$$

(2) Fix ϕ s.t. $d\phi = 0$. If $\partial a = 0$ and $a' = a + \partial c$, then

$$\phi(a') = \phi(a + \partial c) = \phi(a) + \phi(\partial c)$$
$$= \phi(a) + d\phi(c) \cdot (-1)^{|\phi|}$$
$$= \phi(a) + 0.$$

$$(\phi + d\psi)(a) = \phi(a) \pm \psi(\partial a) = \phi(a). \quad //$$

Injection, surjection for homework.

Cup product

Cohomology has a ring structure

$$H^k(X) \otimes H^\ell(R) \to H^{k+\ell}(R).$$

Very cool. We build up to it now.

Notation Note the $(k+\ell)$-simplex

$$\Delta^{k+\ell}$$

has many copies of Δ^k and Δ^ℓ inside it. Let

$$F_k : \Delta^k \to \Delta^{k+\ell}$$

be the embedding induced by the linear map

$$\mathbb{R}^{k+1} \longrightarrow \mathbb{R}^{k+\ell+1}$$
$$x_i \longmapsto x_i .$$

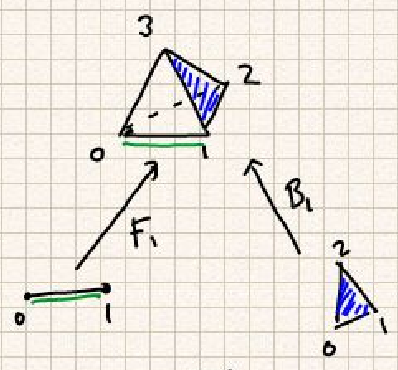
Let

$$B_k : \Delta^\ell \to \Delta^{k+\ell}$$

be the embedding induced by the linear map

$$\mathbb{R}^{\ell+1} \to \mathbb{R}^{k+\ell+1}$$
$$x_i \longmapsto x_{i+k}$$

Ex.



Defn Given $\sigma : \Delta^{k+\ell} \to X$, let

$$\sigma_{0\cdots k} = \sigma \circ F_k$$
$$\sigma_{k\cdots k+\ell} = \sigma \circ B_k .$$

Defn Let

$$C^*(X) \otimes C^*(X) \xrightarrow{\cup} C^*(X)$$

be the map

$$(\phi \cup \psi)(\tau) = \phi(\tau \circ F_k) \cdot \psi(\tau \circ B_k)$$

Rmk Here, $\phi \in C^k(X)$

$\psi \in C^\ell(X)$

and $\phi \cup \psi \in C^{k+\ell}(X)$.

Propn $\cup$ is a map of cochain complexes.

Pf Need to show

$$d(\phi \cup \psi) = d\phi \cup \psi + (-1)^k \phi \cup d\psi .$$

Given $\sigma : \Delta^{k+\ell+1} \to X$,

$$(d\phi \cup \psi)(\sigma) = \sum_{i=0}^{k+1} (-1)^i \phi(\partial_i(\sigma \circ F_{k+1}))\, \psi(\sigma \circ B_{k+1})$$

and

$$(-1)^k (\phi \cup d\psi)(\sigma) = \sum_{j=k}^{k+\ell+1} (-1)^{j-k} \cdot (-1)^k \phi(\sigma \circ F_k)\, \psi(\partial_j(\sigma \circ B_k)) .$$

Note $\partial_{k+1}(\sigma \circ F_{k+1}) = \sigma \circ F_k$

$\partial_k(\sigma \circ B_k) = \sigma \circ B_{k+1}$, so

the $k+1=i$ and $j=k$ terms cancel.

Meanwhile, you can check that

$$(\phi \cup \psi)(\partial\sigma)$$

equals the rest of the summation. //

Cor $\cup$ defines a map

$$H^k(X)\otimes H^\ell(X) \to H^{k+\ell}(X)$$
$$[\phi]\otimes[\psi] \longmapsto [\phi\cup\psi]$$

Pf

If $d\phi=0$, $d\psi=0$, then

$$d(\phi\cup\psi) = d\phi\cup\psi + (-1)^k\phi\cup d\psi$$
$$= 0 + (-1)^k 0$$
$$= 0.$$

So $\phi\cup\psi \in \operatorname{Ker} d^{k+\ell}$.

OTOH, if $\phi'=\phi+d\alpha$, $\psi'=\psi+d\beta$, then

$$\phi'\cup\psi' = \phi\cup\psi + \phi\cup d\beta + d\alpha\cup\psi + d\alpha\cup d\beta.$$

Note
$$\phi\cup d\beta = (-1)^k d(\phi\cup\beta)$$
$$d\alpha\cup\psi = d(\alpha\cup\psi)$$
$$d\alpha\cup d\beta = d(\alpha\cup d\beta).$$

So $\phi'\cup\psi' = \phi\cup\psi + d(\text{something})$ //

Prop'n $\cup : H^*(X)\otimes H^*(X)\to H^*(X)$ is associative. If $X\neq\emptyset$, it is unital.

Pf Note

$$\big(\psi_1\cup(\psi_2\cup\psi_3)\big)(\sigma) = \psi_1(\sigma\circ F_{k_1})\,(\psi_2\cup\psi_3)(\sigma\circ B_{k_2})$$
$$= \psi_1(\sigma_{0\ldots k_1})\,\psi_2(\sigma_{k_1\ldots k_2})\,\psi_3(\sigma_{k_2\ldots k})$$
$$= (\psi_1\cup\psi_2)(\sigma\circ F_{k_2})\,\psi_3(\sigma\circ B_{k_2})$$
$$= \big((\psi_1\cup\psi_2)\cup\psi_3\big)(\sigma).$$

Setting

$$\phi(\sigma) = 1 \quad \forall\, \sigma\in C_0(X),$$
$$\phi\in C^0(X),$$

we see

$$(\phi\cup\psi)(\tau) = \phi(\tau\circ F_0)\,\psi(\tau\circ B_0)$$
$$= \psi(\tau).$$

Likewise, $\psi\cup\phi = \phi$. //

Thm
$$[\phi]\cup[\psi] = (-1)^{k\ell}[\psi]\cup[\phi]$$
$\forall\ [\phi]\in H^k(X),\ [\psi]\in H^\ell(X)$.

Pf Ommitted for now //

Cor $H^{\cdot}(X)$ is a graded, graded-commutative ring.

If $X\neq\phi$, it is unital.

Prop'n Let $f: X\to Y$ be C^0. Then $f^*: H^*(Y)\to H^*(X)$ is a map of rings.

Pf.
$$(f^*(\phi\cup\psi))(\sigma) = \phi\cup\psi\,(f\circ\sigma)$$
$$= \phi(f\circ\sigma\circ F_k)\,\psi(f\circ\sigma\circ B_k)$$
$$= f^*\phi(\sigma\circ F_k)\,f^*\psi(\sigma\circ B_k)$$
$$= (f^*\phi\cup f^*\psi)(\sigma).\ //$$

Cor H^* defines a functor

$$\text{Spaces}^{op} \to \text{GCRings}.$$

Ex If $X = S^n$,

$$H^*(X) \cong H^0(X) \oplus H^n(X).$$

If 1 is the function
$$\sigma \mapsto 1 \quad \forall \sigma \in C_0,$$
and $f \in H^n(X)$, we have
$$f^2 = 0, \quad 1 \cdot f = f \cdot 1 = f.$$
Very boring ring. Isomorphic to
$$\mathbb{Z}[f]/_{f^2}, \quad |f| = n.$$

Ex If $X = S^1 \times S^1$, we have

$$H^*(X) \cong H^0(X) \oplus H^1(X) \oplus H^2(X)$$
$$\cong \mathbb{Z} \oplus \mathbb{Z}^2 \oplus \mathbb{Z}$$

as a group.

It turns out we can find generators
$$\alpha, \beta \in H^1(X)$$
s.t.
$$\alpha \cup \beta \neq 0,$$
and $\alpha \cup \beta$ generates $H^2(X)$.

Hence
$$H^*(X) \cong \mathbb{Z}[\alpha, \beta] / (\alpha^2, \beta^2, \alpha\beta - \beta\alpha)$$
w/ $|\alpha| = |\beta| = 1$.

Relative H^*

The whole point of relative homology was exploiting the LES.

So we want the same from H^*.

Defn Let $A \subset X$. Then the cochain complex of relative cochains,
$$C^*(X, A) \subset C^*(X)$$
is the subcomplex of those
$$\phi : C_*(X) \to \mathbb{Z}$$
s.t.
$$\phi|_{C_*(A)} \equiv 0.$$

I.e., $\phi \in C^*(X, A) \iff \phi$ vanishes on A.

Note if $\phi \in C^*(X, A)$, then
$$d\phi(\sigma) = \phi(\partial\sigma) = 0$$
$\forall \sigma$ s.t. $\partial\sigma \in C_*(A)$. In particular, $\forall \sigma \in C_{*+1}(A)$. (ie, C^* is closed under d).

Prop $\exists$ LES of H^* groups

$$\cdots \to H^n(X,A) \to H^n(X) \to H^n(A) \xrightarrow{\delta} H^{n+1}(X,A) \to H^{n+1}(X) \to \cdots$$

δ increases degree!

Pf We have SES
$$0 \to C^n(X,A) \to C^n(X) \to C^n(A) \to 0$$
$\forall n$. So just flip LES for H_* upside down.