

# Relative Cohomology

(1)

The whole point of relative  $H_*$  was the LES. Let's develop the same for  $H^*$ .

## Homology

$$A \hookrightarrow X \quad \text{subspace}$$

}  
}

$$C_n(A) \hookrightarrow C_n(X) \quad \text{injection of abelian grps}$$

}  
}

$$C_n(A) \hookrightarrow C_n(X) \rightarrow \frac{C_n(X)}{C_n(A)} =: C_n(X, A)$$

}  
}

LES of  $H_*$ .

Mimicking for  $H^*$ , we have

(2)

$$\begin{array}{ccc}
 A & \hookrightarrow & X \\
 & \downarrow & \\
 C^n(A) & \hookleftarrow & C^n(X)
 \end{array}$$

surjection! So to construct a SES, we want to take its kernel.

$$\begin{array}{ccc}
 C^n(A) & \hookleftarrow & C^n(X) \hookleftarrow C^n(X, A)
 \end{array}$$

Since  $d[C^n(X, A)] \subset C^{n+1}(X, A)$ ,  $C^\bullet(X, A)$  defines a cochain complex.

LES of  $H^0$ .

Defn Let  $C^n(X, A) \subset C^n(X)$  be the

subgroup of cochains  $\phi$  s.t.  $\phi(\sigma) = 0$

$\forall \sigma \in C_n(A)$ . Since  $d\phi(\tau) = \pm \phi(\partial\tau) = 0$  if  $\tau \in C_{n+1}(A)$ , this defines a cochain complex  $C^\bullet(X, A)$ .

Def The  $n^{\text{th}}$  relative cohomology group of the pair  $(X, A)$  is

$$H^n(X, A) := H^n(C^*(X, A))$$

Thm  $\forall A \subset X, \exists$  LES

$$\begin{array}{ccccccc}
\cdots & \longrightarrow & H^n(X, A) & \longrightarrow & H^n(X) & \longrightarrow & H^n(A) \\
& & & & & & \searrow \delta \\
& & & & & & H^{n+1}(X, A) \longrightarrow \cdots
\end{array}$$

Remark The connecting map  $\delta: H^n(A) \rightarrow H^{n+1}(X, A)$

increases degree. Given  $\phi \in C^n(A)$  s.t.

$d\phi = 0$ , let  $\phi_x: C^n(X) \rightarrow \mathbb{Z}$  be a cochain so  $\phi_x|_{C_n(A)} = \phi$ .

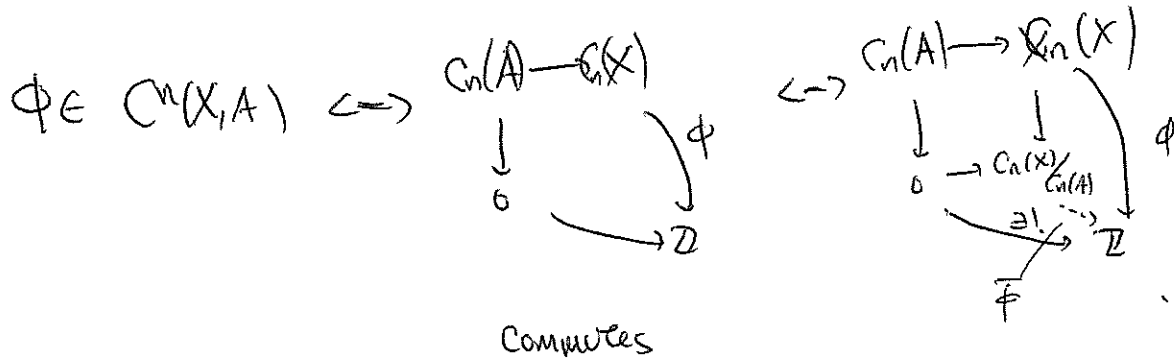
Then  $\delta(\phi) = [d\phi_x]$ .

Ex:  $d\phi_x(\sigma) = \phi_x(\partial\sigma)$  if  $\partial\sigma \subset A$   
if  $\partial\sigma \not\subset A = 0 + \phi(\partial\sigma)$

Prop As a cochain complex,

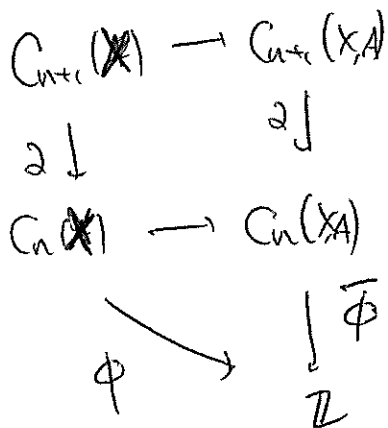
$$C^0(X, A) \cong \text{hom}^0(C_0(X, A), \mathbb{Z})$$

Pf As abelian groups,



So we have a bijection  $C^n(X, A) \cong \text{hom}(C_n(X, A), \mathbb{Z})$ .

As for differential,



Commutative. //

Cor  $\exists$  SES, natural,

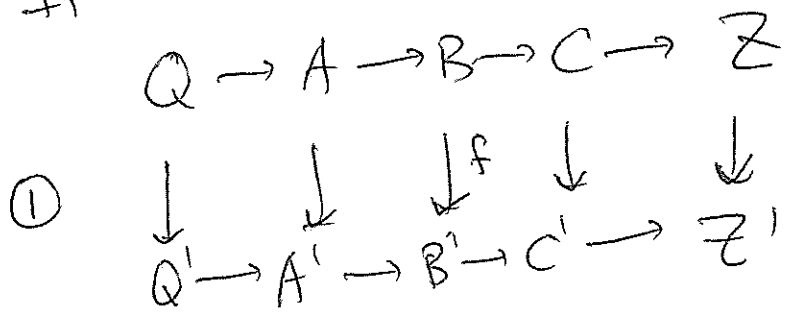
$$0 \rightarrow \text{Ext}(H_{n+1}(X, A), \mathbb{Z}) \rightarrow H^n(X, A) \rightarrow \text{hom}(H_n(X, A), \mathbb{Z}) \rightarrow 0$$

This splits, unnaturally.

Here's an algebraic lemma we haven't explicitly stated yet, but is very useful.

Lemma (Five Lemma).

If



is a comm. diagram,

② the top and bottom rows are exact, and

③ every vertical arrow aside from  $f$  is known to be  $\cong$ , then

$f: B \rightarrow B'$  is an isomorphism.

(6)

Lemma Let  $A, B$  be chain complexes,

Let  $f, g: A \rightarrow B$  be two chain maps that are chain homotopic.

Then the induced maps

$$f^*, g^*: \text{hom}(B, \mathbb{Z}) \rightarrow \text{hom}(A, \mathbb{Z})$$

are chain homotopic.

Pf. Let  $H_n: A_n \rightarrow B_{n+1}$  s.t.

$$\partial_{n+1}^B H + H \partial_n^A = f_n - g_n.$$

Consider  $Q^{n+1}: \text{hom}(B_{n+1}, \mathbb{Z}) \xrightarrow{\circ H_n} \text{hom}(A_n, \mathbb{Z})$   
 $\phi \longmapsto (-1)^n \phi \circ H_n.$

$$\begin{aligned} \text{Then } (d^n Q^{n+1} + Q^{n+2} d^{n+1})(\phi) &= (-1)^n d^n(\phi \circ H_n) + (-1)^{n+1} Q^{n+2}(\phi \circ \partial_{n+2}) \\ &= (-1)^n \phi \circ H_n \circ \partial_{n+1} + (-1)^{n+1} (\phi \circ \partial_{n+2} \circ H_{n+1}) \\ &= \phi(H_n \partial_{n+1} + \partial_{n+2} H_{n+1}) \\ &= \phi(f_{n+1} - g_{n+1}) \\ &= f^* \phi - g^* \phi \\ &= (f^* - g^*) \phi. \quad // \end{aligned}$$

Cor ① If  $f_0 \sim f_1: X \rightarrow Y$ ,  
 then  $f_1^* = f_0^*$  on  $H^*$ .

② If  $f_0, f_1: (X, A) \rightarrow (Y, B)$   
 are homotopic as maps of pairs,  
 then  
 $f_0^* = f_1^*: H^n(Y, B) \rightarrow H^n(X, A)$ .

Pf ① since  $f_0 \sim f_1 \Rightarrow f_0 \sim f_1$  as chain maps  $C_n X \rightarrow C_n Y$   
 $\Rightarrow f_0^* \sim f_1^*$  as cochain maps  $C^n Y \rightarrow C^n X$   
 $\Rightarrow f_0^*, f_1^*$  induce same maps on  $H^n$ .

② Note the homotopy  $Q$  restricts to a  
 homotopy on  $C^n(X, A) \subset C^n(X)$ . //

Thm (Excision)

If  $A \subset X$  and  $\bar{A} \subset U$  for  $U$  open,  
 then the inclusion

$$(X-A, U-A) \hookrightarrow (X, U)$$

induces an  $\cong$

$$H^*(X, U) \xrightarrow{\cong} H^*(X-A, U-A)$$

Prf. By excision for homology, we know  
 the inclusion induces  $\cong$  on  $H_0$ .

By univ coeff thm, we have a comm. diagram

$$\begin{array}{ccccccc}
 0 & \longrightarrow & \text{Ext}(H_{n-1}(X-A, U-A), \mathbb{Z}) & \longrightarrow & H^n(X-A, U-A) & \longrightarrow & \text{hom}(H_n(X-A, U-A), \mathbb{Z}) \longrightarrow 0 \\
 \uparrow & & \uparrow \cong \text{excision for } H_0 & & \uparrow & & \uparrow \text{excision for } H_0 \\
 0 & \longrightarrow & \text{Ext}(H_{n-1}(X, U), \mathbb{Z}) & \longrightarrow & H^n(X, U) & \longrightarrow & \text{hom}(H_n(X, U), \mathbb{Z}) \longrightarrow 0
 \end{array}$$

so by five lemma, middle map is  $\cong$  as well.





Cor If  $A \subset X$  is closed,  
and  $A \subset U$  for some open  $U$   
that def retracts onto  $A$ , then

$$H^n(X, A) \cong H^n(X/A, *)$$

Pf Inclusions  $(X, A) \hookrightarrow (X, U)$ ,  $(X-A, U-A) \hookrightarrow (X, U)$  induce

maps

$$H^n(X, A) \xleftarrow{(4)} H^n(X, U) \xrightarrow{(2)} H^n(X-A, U-A)$$

$$H^n(X/A, *) \xleftarrow{(4)} H^n(X/A, U/A) \xrightarrow{(2)} H^n(X/A - U/A, U/A - A/A)$$

$g^* \uparrow (5) \cong$  by commutativity

$g^* \uparrow (3) \cong$  by commutativity

$g^* \uparrow (1) \cong$  since  $X-A \downarrow \text{quotient}$   
 $X/A - A/A$   
is homeo.

$\cong$  since  $V$  retracts onto  $A$ .

$\cong$  by excision



Prop If  $X = \coprod_{\alpha} X_{\alpha}$ , and  $A_{\alpha} \subset X_{\alpha}$ ,  $A = \coprod_{\alpha} A_{\alpha}$ ,  
 then

$$H^n(X, A) \cong \prod_{\alpha} H^n(X_{\alpha}, A_{\alpha})$$

↳ direct product, not sum.

Pf Assume  $A_{\alpha} = \emptyset$ , so we're dealing w/ usual cohomology.

Case for  $A_{\alpha} \neq \emptyset$  is the same.

$$\begin{aligned} C^n(\coprod_{\alpha} X_{\alpha}) &= \text{hom}^n(\bigoplus_{\alpha} C_n(X_{\alpha}), \mathbb{Z}) \\ &= \prod_{\alpha} \text{hom}^n(C_n(X_{\alpha}), \mathbb{Z}). \end{aligned}$$

We see  $(\phi_{\alpha}) \in C^n(\coprod_{\alpha} X_{\alpha})$  is closed  $\Leftrightarrow d_{\alpha} \phi_{\alpha} = 0 \forall \alpha$ .

So the map  $\prod_{\alpha} H^n(X_{\alpha}) \rightarrow H^n(\coprod_{\alpha} X_{\alpha})$  is a surjection.  
 $\alpha \quad [(\phi_{\alpha})] \mapsto [(\phi_{\alpha})]$

OTOH,  $[(\phi_{\alpha})] = 0 \Leftrightarrow (\phi_{\alpha}) = d(\psi_{\alpha}) \Leftrightarrow \phi_{\alpha} = d\psi_{\alpha} \forall \alpha \Rightarrow [(\phi_{\alpha})] = (0)$ .

So it's injective. //

Def (Eilenberg-Steenrod Axioms) Let  $\text{Pairs}$  be category of pairs of spaces

A cohomology theory is a sequence of functors

$$K^n: \text{Pairs}^{\text{op}} \rightarrow \text{AbGroups}, \quad n \in \mathbb{Z}$$

together w/ natural transformations

$$\delta: K^{n-1}(A, \phi) \rightarrow K^n(X, A), \quad n \in \mathbb{Z}$$

s.t.

(1) (Homotopy)

$$f \sim g: (X, A) \rightarrow (Y, B) \Rightarrow K^n(f) = K^n(g) \quad \forall n.$$

(2) (Excision). If  $A \subset U$ ,  $U \subset X$  open, then inclusion induces  $\cong$

$$K^n(X, U) \cong K^n(X - A, U - A) \quad \forall n$$

(3) (Additivity). If  $X = \coprod_{\alpha} X_{\alpha}$ ,  $A_{\alpha} \subset X_{\alpha}$ ,  $A = \coprod_{\alpha} A_{\alpha}$ ,

then

$$K^n(X, A) \cong \prod_{\alpha} K^n(X_{\alpha}, A_{\alpha}) \quad \forall n$$

(4) (Exactness) The sequence

$$\rightarrow K^n(X, A) \rightarrow K^n(X, \phi) \rightarrow K^n(A, \phi) \xrightarrow{\delta} K^{n+1}(X, A) \rightarrow \dots$$

is exact. //