Relative Cohomology

The whole point of relative $H_\bullet$ was the LES. Lets develop the same for $H^\bullet$.

Homology

$A \hookrightarrow X$ subspace

$\rightsquigarrow$

$C_n(A) \hookrightarrow C_n(X)$ injection of abelian grps

$\rightsquigarrow$

$$C_n(A) \hookrightarrow C_n(X) \to C_n(X)/C_n(A) =: C_n(X, A)$$

$\rightsquigarrow$

LES of $H_\bullet$.

Mimicking for H^*, we have

$$A \hookrightarrow X$$

$$C^n(A) \leftarrow C^n(X)$$

surjection! So to construct a SES, we want to take its kernel.

$$C^n(A) \leftarrow C^n(X) \leftarrow C^n(X,A).$$

Since $d(C^n(X,A)) \subset C^{n+1}(X,A)$, $C^\bullet(X,A)$ defines a cochain complex.

LES of $H^\bullet$.

<u>Defn</u> Let $C^n(X,A) \subset C^n(X)$ be the subgrp of cochains ϕ s.t. $\phi(\sigma) = 0$ $\forall\, \sigma \in C_n(A)$. Since $d\phi(\tau) = \pm\phi(\partial\tau) = 0$ if $\tau \in C_{n+1}(A)$, this defines a cochain complex $C^\bullet(X,A)$.

Defn The n^{th} relative cohomology group of the pair (X,A) is

$$H^n(X,A) := H^n(C^\bullet(X,A)).$$

Thm $\forall\ A\subset X$, $\exists$ LES

$$\cdots \longrightarrow H^n(X,A) \longrightarrow H^n(X) \longrightarrow H^n(A) \xrightarrow{\delta} H^{n+1}(X,A) \longrightarrow \cdots$$

Rmk The connecting map $\delta: H^n(A) \to H^{n+1}(X,A)$ increases degree. Given $\phi\in C^n(A)$ s.t. $d\phi=0$, let $\phi_X: C_n(X)\to\mathbb{Z}$ be a cochain so $\phi_X|_{C_n(A)}=\phi$. Then $\delta(\phi)=[d\phi_X]$.

Ex: $d\phi_X(\sigma)=\phi_X(\partial\sigma)$

$= 0 + \phi(\partial_i\sigma)$ if $\partial_j\sigma\not\subset A$ · if $\partial_i\sigma\subset A$

<u>Prop</u> As a cochain complex,

$$C^\bullet(X,A) \cong \hom^\bullet(C_\bullet(X,A), \mathbb{Z}).$$

<u>Pf</u> As abelian groups,

$$\phi \in C^n(X,A) \iff \begin{tikzcd} C_n(A) \arrow[r] \arrow[d] & C_n(X) \arrow[d, bend left, "\phi"] \\ 0 \arrow[r] & \mathbb{Z} \end{tikzcd} \iff \begin{tikzcd} C_n(A) \arrow[r] \arrow[d] & C_n(X) \arrow[d] \arrow[dd, bend left, "\phi"] \\ 0 \arrow[r] \arrow[dr] & C_n(X)/C_n(A) \arrow[d, dashed, "\exists! \bar\phi"] \\ & \mathbb{Z} \end{tikzcd}$$

commutes

So we have a bijection $C^n(X,A) \xrightarrow{\cong} \hom(C_n(X,A), \mathbb{Z})$.

As for differential,

$$\begin{tikzcd} C_{n+1}(X) \arrow[r] \arrow[d, "\partial"] & C_{n+1}(X,A) \arrow[d, "\partial"] \\ C_n(X) \arrow[r] \arrow[dr, "\phi"'] & C_n(X,A) \arrow[d, "\bar\Phi"] \\ & \mathbb{Z} \end{tikzcd}$$

commutes. //

<u>Cor</u> $\exists$ SES, natural,

$$0 \to \mathrm{Ext}(H_{n-1}(X,A), \mathbb{Z}) \to H^n(X,A) \to \hom(H_n(X,A), \mathbb{Z}) \to 0.$$

This splits, unnaturally.

Here's an algebraic lemma we haven't explicitly stated yet, but is very useful.

<u>Lemma</u> (Five Lemma).

If

① $$\begin{tikzcd} Q \arrow[r] \arrow[d] & A \arrow[r] \arrow[d] & B \arrow[r] \arrow[d, "f"] & C \arrow[r] \arrow[d] & Z \arrow[d] \\ Q' \arrow[r] & A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & Z' \end{tikzcd}$$

is a comm. diagram,

② the top and bottom rows are exact, and

③ every vertical arrow aside from f is known to be $\cong$, then

$f : B \longrightarrow B'$ is an isomorphism.

Lemma Let $A_\bullet$, $B_\bullet$ be chain complexes,

Let $f, g : A_\bullet \to B_\bullet$ be two chain maps that are chain homotopic.

Then the induced maps

$$f^*, g^* : \hom^\bullet(B_\bullet, \mathbb{Z}) \to \hom^\bullet(A_\bullet, \mathbb{Z})$$

are chain homotopic.

Pf. Let $H_n : A_n \to B_{n+1}$ s.t.

$$\partial^B_{n+1} H + H \partial^A_n = f_n - g_n.$$

Consider $Q^{n+1} : \hom(B_{n+1}, \mathbb{Z}) \xrightarrow{\circ H_n} \hom(A_n, \mathbb{Z})$

$$\phi \longmapsto (-1)^n \phi \circ H_n.$$

Then

$$\begin{aligned}
(d^n Q^{n+1} + Q^{n+2} d^{n+1})(\phi) &= (-1)^n d^n(\phi \circ H_n) + (-1)^{n+1} Q^{n+2}(\phi \circ \partial_{n+2}) \\
&= (-1)^{2n} \phi \circ H_n \circ \partial_{n+1} + (-1)^{2(n+1)} (\phi \circ \partial_{n+2} \circ H_{n+1}) \\
&= \phi(H_n \partial_{n+1} + \partial_{n+2} H_{n+1}) \\
&= \phi(f_{n+1} - g_{n+1}) \\
&= f^* \phi - g^* \phi \\
&= (f^* - g^*)\phi.
\end{aligned}$$

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Cor ① If $f_0 \sim f_1 : X \to Y$,

then $f_1^* = f_0^*$ on H^*.

② If $f_0, f_1 : (X, A) \to (Y, B)$

are homotopic as maps of pairs,

then

$$f_0^* = f_1^* : H^n(Y, B) \longrightarrow H^n(X, A).$$

Pf ① since $f_0 \sim f_1 \Rightarrow f_0 \sim f_1$ as chain maps $C_n X \to C_n Y$

$\Rightarrow f_0^* \sim f_1^*$ as cochain maps $C^n Y \to C^n X$

$\Rightarrow f_0^*, f_1^*$ induce same map on H^n.

② Note the homotopy Q restricts to a homotopy on $C^n(X, A) \subset C^n(X)$. //

<u>Thm</u> (Excision)

If $A \subset X$ and $\bar{A} \subset U$ for U open,

then the inclusion

$$(X - A, U - A) \hookrightarrow (X, U)$$

induces an $\cong$

$$H^*(X, U) \longrightarrow H^*(X - A, U - A).$$

<u>Pf.</u> By excision for homology, we know

the inclusion induces $\cong$ on H_*.

By univ coeff thm, we have a comm. diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Ext}(H_{n-1}(X-A, U-A), \mathbb{Z}) \arrow[r] & H^n(X-A, U-A) \arrow[r] & \hom(H_n(X-A, U-A), \mathbb{Z}) \arrow[r] & 0 \\
0 \arrow[u] \arrow[r] & \mathrm{Ext}(H_{n-1}(X, U), \mathbb{Z}) \arrow[u, "\cong\ \text{excision for } H_*"'] \arrow[r] & H^n(X, U) \arrow[u] \arrow[r] & \hom(H_n(X, U), \mathbb{Z}) \arrow[u, "\cong\ \text{excision for } H_*"'] \arrow[r] & 0 \arrow[u]
\end{tikzcd}$$

so by five lemma, middle map is $\cong$ as well. //

Cor If $A \subset X$ is closed, and $A \subset U$ for some open U that def retracts onto A, then

$$H^n(X, A) \cong H^n(X/A, *).$$

Pf. Inclusions $(X,A) \hookrightarrow (X,U)$, $(X-A, U-A) \hookrightarrow (X,U)$ induce maps

$$\begin{tikzcd}
H^n(X,A) & H^n(X,U) \arrow[l, "④"'] \arrow[r, "②"] & H^n(X-A, U-A) \\
H^n(X/A, *) \arrow[u, "q^*", "⑤ \cong \text{ by commutativity}"'] & H^n(X/A, U/A) \arrow[l, "④"] \arrow[r, "②"'] \arrow[u, "q^*", "③ \cong \text{ by commutativity}"'] & H^n(X/A - A/A, U/A - A/A) \arrow[u, "q^*", "① \cong"']
\end{tikzcd}$$

① $\cong$ since $X-A \xrightarrow{\text{quotient}} X/A - A/A$ is homeo.

④ $\cong$ since U retracts onto A.

② $\cong$ by excision.

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<u>Prop</u> If $X = \coprod_\alpha X_\alpha$, and $A_\alpha \subset X_\alpha$, $A = \coprod_\alpha A_\alpha$,

then

$$H^n(X, A) \cong \prod_\alpha H^n(X_\alpha, A_\alpha)$$

direct product, not sum.

<u>Pf</u> Assume $A_\alpha = \phi$, so we're dealing w/ usual cohomology.

Case for $A_\alpha \neq \phi$ is the same.

$$C^n(\coprod_\alpha X_\alpha) = \hom^n(\bigoplus_\alpha C_n(X_\alpha), \mathbb{Z})$$

$$= \prod_\alpha \hom^n(C_n(X_\alpha), \mathbb{Z}).$$

We see $(\phi_\alpha) \in C^n(\coprod_\alpha X_\alpha)$ is closed $\Leftrightarrow d_\alpha \phi_\alpha = 0 \ \forall \alpha$.

So the map $\prod_\alpha H^n(X_\alpha) \longrightarrow H^n(\coprod X_\alpha)$ is a surjection.

$$([\phi_\alpha]) \longmapsto [(\phi_\alpha)]$$

OTOH, $[(\phi_\alpha)] = 0 \Leftrightarrow (\phi_\alpha) = d(\psi_\alpha) \Leftrightarrow \phi_\alpha = d\psi_\alpha \ \forall \alpha \Leftrightarrow ([\phi_\alpha]) = ([0])$.

So it's injective. //

<u>Defn</u> (Eilenberg-Steenrod Axioms) Let Pairs be category of pairs of spaces

A cohomology theory is a sequence of functors

$$K^n : \text{Pairs}^{op} \longrightarrow \text{AbGroups}, \quad n \in \mathbb{Z}$$

together w/ natural transformations

$$\delta : K^{n-1}(A, \phi) \longrightarrow K^n(X, A), \quad n \in \mathbb{Z}$$

s.t.

(1) (Homotopy)

$$f \sim g : (X, A) \to (Y, B) \Rightarrow K^n(f) = K^n(g) \quad \forall n.$$

(2) (Excision). If $\bar{A} \subset \mathcal{U}$, $\mathcal{U} \subset X$ open, then inclusion induces $\cong$

$$K^n(X, \mathcal{U}) \cong K^n(X - A, \mathcal{U} - A) \quad \forall n$$

(3) (Additivity). If $X = \coprod_\alpha X_\alpha$, $A_\alpha \subset X_\alpha$, $A = \coprod_\alpha A_\alpha$,

then

$$K^n(X, A) \cong \prod_\alpha K^n(X_\alpha, A_\alpha) \quad \forall n$$

(4) (Exactness) The sequence

$$\longrightarrow K^n(X, A) \to K^n(X, \phi) \to K^n(A, \phi) \xrightarrow{\delta} K^{n+1}(X, A) \to \cdots$$

is exact. //