Defn (Eilenberg–Steenrod Axioms) Let Pairs be category of pairs of spaces

A cohomology theory is a sequence of functors

$$K^n : \mathrm{Pairs}^{op} \longrightarrow \mathrm{AbGroups}, \quad n \in \mathbb{Z}$$

together w/ natural transformations

$$\delta : K^{n-1}(A, \phi) \longrightarrow K^n(X, A), \quad n \in \mathbb{Z}$$

$$\begin{tikzcd} (X,A) \arrow[d, mapsto] & \mathrm{Pairs}^{op} \arrow[r, "K^n"] \arrow[d] & \mathrm{AbGps} \\ (A,\phi) & \mathrm{Pairs}^{op} \arrow[ur, "K^{n-1}"'] \arrow[u, phantom, shift left=8ex, "\Uparrow\delta"] & \end{tikzcd}$$

i.e.,

$$\begin{tikzcd} \mathrm{Pairs}^{op} \arrow[r, bend left, "K^n"] \arrow[r, bend right, "K^{n-1}(A,\phi)"'] \arrow[r, phantom, "\Uparrow\delta"] & \mathrm{AbGps} \end{tikzcd}$$

s.t.

(1) (Homotopy)

$$f \sim g : (X, A) \longrightarrow (Y, B) \Rightarrow K^n(f) = K^n(g) \quad \forall n.$$

(2) (Excision). If $\bar{A} \subset \mathcal{U}$, $\mathcal{U} \subset X$ open, then inclusion induces $\cong$

$$K^n(X, \mathcal{U}) \cong K^n(X - A, \mathcal{U} - A) \quad \forall n$$

(3) (Additivity). If $X = \coprod_\alpha X_\alpha$, $A_\alpha \subset X_\alpha$, $A = \coprod_\alpha A_\alpha$,

then

$$K^n(X, A) \cong \prod_\alpha K^n(X_\alpha, A_\alpha) \quad \forall n$$

(4) (Exactness) The sequence

$$\longrightarrow K^n(X, A) \longrightarrow K^n(X, \phi) \longrightarrow K^n(A, \phi) \xrightarrow{\delta} K^{n+1}(X, A) \longrightarrow \cdots$$

is exact. //

Thm H° is a cohomology theory.

Pf Last class.

Thm Let H°, K° be cohomology theories, and $u^n : H^n \to K^n$ natural transformations compatible w/ δ_H and δ_K. If u induces an $\cong$

$$H^n(pt, \phi) \to K^n(pt, \phi)$$

$\forall n$, then $u^n : H^n(X,A) \cong K^n(X,A)$ $\forall n$.

Pf Hmwk, for finite CW complexes. General case requires a limit argument. //

[H° w/ coefficients!]

Fix a ring R. Then $H^\circ(-;R)$ is another cohom. theory, constructed as follows:

$$\begin{tikzcd}
\text{Spaces} \arrow[r] & \text{ssSet} \arrow[r] & \text{Chain}_R \arrow[r, "\text{contravariant}"] & \text{Cochain}_R \arrow[r, "H^\circ"] & R\text{-mod} \arrow[r, "\text{forget}"] & \text{abelian groups} \\
X \arrow[r, mapsto] & \text{Sing}(X) & C_\bullet(X;R) & C^n(X;R) & &
\end{tikzcd}$$

$$\text{Sing}(X) = \text{Maps}(\Delta^\bullet, X)$$

$$C_\bullet(X;R) = \text{Free } R\text{-module } \bigoplus_{\text{Maps}(\Delta^n, X)} R \cong C_\bullet(X) \otimes R$$

$$C^n(X;R) = \text{hom}_{R\text{-mod}}(C_\bullet(X;R), R) \cong \text{hom}_{\mathbb{Z}\text{-mod}}(C_\bullet(X), R)$$

This line makes sense → for R not a ring, and just an abelian group, π. This gives $H_\bullet$ and $H^\bullet$ w/ coefficients in any abelian group π.

Künneth turns out to be compatible w/ R-module structure:

$$0 \longrightarrow \underset{\mathbb{Z}\circlearrowleft}{} H_n(X) \otimes R \underset{\text{action}}{\circlearrowleft R} \longrightarrow H_n(X;R) \underset{\text{action}}{\circlearrowleft R} \longrightarrow \operatorname{Tor}_1(H_{n-1}(X), R) \underset{\text{action}}{\circlearrowleft R} \longrightarrow 0$$

We also have a univ. coeff thm for R-modules.

<u>Defn</u> Let M, N be R-modules.
Then a free resolution for M is a chain complex $F_\bullet$, s.t. $F_n = R^{\oplus k_n}$ is a free R-module $\forall n$, together w/ a map $F_0 \to M_0$ inducing an isomorphism

$$H_0(F_\bullet) \cong M_0$$
$$H_n(F_\bullet) \cong 0 \quad \forall\, n \neq 0$$

Ext^n is the cohomology of the hom complex.
i.e.,

$$\operatorname{Ext}^n_R(M, N) := H^n(\underbrace{\hom^\bullet_R}_{\text{homs of R-modules}}(F_\bullet, N)).$$

Thm (Univ Coeff Thm)

Let R be a principal ideal domain. (Ex $\mathbb{Z}$, or any field).
Let $A_\bullet$ be a chain complex s.t. A_n is a free R-module $\forall n$. Then $\exists$ natural SES

$$0 \to \mathrm{Ext}^1_R(H_{n-1}(A), R) \to H^n(\hom^\bullet(A_\bullet, R)) \to \hom(H_n(A_\bullet), R) \to 0$$

$\forall n$. This splits (unnaturally).

This computes $H^n(X;R)$ as R-modules.

Relative cup product

Let (X,A) be a pair of spaces.

Prop'n If $\varphi, \psi \in C^k(X,A), C^\ell(X,A)$ respectively. ~~Then~~

$$\varphi \cup \psi \in C^{k+\ell}(X,A).$$

Pf: If $\sigma \in C_{k+\ell}(A)$, then $\sigma \circ B_k, \sigma \circ F_k \in C_\bullet(A)$.
So $\varphi \cup \psi(\sigma) = \sum_k \varphi(\sigma \circ F_k)\psi(\sigma \circ B_\ell) = 0$. //

Since $C^k(X,A) \subset C^k(X)$, all the properties of cup product stay the same.

Prop'n The map

$$\cup : H^\bullet(X,A) \otimes H^\bullet(X,A) \to H^\bullet(X,A)$$
$$[\varphi] \otimes [\psi] \mapsto [\varphi \cup \psi]$$

is graded commutative and associative.

Likewise, if $f(A) \subset B \subseteq Y$,

then $\phi \in C^k(Y,B) \Rightarrow f^*\phi$ vanishes on A.

So $f: (X,A) \to (Y,B)$ induces a map

$$f^*: H^\bullet(Y,B) \to H^\bullet(X,A).$$

This is again a map of rings.

<u>Prop</u> $H^\bullet(\ )$ defines a functor

$$\mathrm{Pair}^{op} \longrightarrow \mathrm{GrRings}.$$

Note $H^\bullet(X,A)$ is not unital unless $A = \phi$, where $H^\bullet(X,A) \cong H^\bullet(X)$.

More generally, if $A_1, A_2 \subset X$,

let $C^n(X; A_1 + A_2)$ be the cochains that vanish on $C_n(A_1) + C_n(A_2) \subset C_n(X)$.

Then $\phi \in C^k(X, A_1)$, $\psi \in C^k(X, A_2)$

$$\Longrightarrow \phi \cup \psi \in C^k(X, A_1 + A_2).$$

So we have a map

$$H^\bullet(X,A_1) \otimes H^\bullet(X,A_2) \to H^\bullet(C^\bullet(X, A_1 + A_2)).$$

<u>Thm</u> The map

$$C^\bullet(X, A_1 + A_2) \leftarrow C^\bullet(X, A_1 \cup A_2)$$

induces an $\cong$

$$H^\bullet(X, A_1 + A_2) \cong H^\bullet(X, A_1 \cup A_2).$$

Cor Relative cup product

induces a map

$$H^\bullet(X,A_1)\otimes H^\bullet(X,A_2)\longrightarrow H^\bullet(X,A_1\cup A_2).$$

This is a map of rings.

(LHS is $\otimes$ of two gr comm rings, so it's a comm ring via $(a\otimes b)\cdot(c\otimes d):=(-1)^{|c||b|}ac\otimes bd$.)

Cross Product.

So much structure!

There's also a map

$$\times: H^*(X)\otimes H^\bullet(Y)\longrightarrow H^\bullet(X\times Y)$$

and the relative version is

$$\times: H^\bullet(X,A)\otimes H^\bullet(Y,B)\longrightarrow H^\bullet(X\times Y,\ A\times Y\cup X\times B).$$

Defined as follows:

Let $p_1: X\times Y\to X$, $p_2: X\times Y\to Y$ be the projections. Then $\times$ is the composite

$$H^\bullet(X)\otimes H^\bullet(Y)\xrightarrow{p_1^*\otimes p_2^*} H^\bullet(X\times Y)\otimes H^\bullet(X\times Y)\xrightarrow{\cup} H^\bullet(X\times Y).$$

$$H^\bullet(X,A)\otimes H^\bullet(Y,B)\xrightarrow{p_1^*\otimes p_2^*} H^\bullet(X\times Y,A\times Y)\otimes H^\bullet(X\times Y,X\times B)\xrightarrow{\cup} H^\bullet(X\times Y,\ A\times Y\cup X\times B).$$

Prop'n Consider the map

$$\begin{aligned} X &\xrightarrow{\Delta} X\times X \\ x &\longmapsto (x,x) \end{aligned}$$

The diagram

$$\begin{tikzcd} H^\bullet(X)\otimes H^\bullet(X) \arrow[rr, "\cup"] \arrow[dr, "\times"'] & & H^\bullet(X) \\ & H^\bullet(X\times X) \arrow[ur, "\Delta^*"'] & \end{tikzcd}$$

commutes.

Rmk This is important philosophically.
If one has some other (non-$\cup$-dependent) definition for the cross product $\times$, this tells us we can define $\cup$ to be $\Delta^* \circ \times$.
If you're looking for a reason why every space gives rise to some ring structure, this is it:
Every space has a map $\Delta: X \to X\times X$.

Pf
$$\begin{aligned} (\Delta^*\circ \times)([\phi]\otimes[\psi]) &= \Delta^*\left(p_1^*[\phi]\cup p_2^*[\psi]\right) \\ &= \Delta^* p_1^*[\phi] \cup \Delta^* p_2^*[\psi] \quad \text{since } f^* \text{ is a ring map } \forall \text{ cts } f. \\ &= id^*[\phi]\cup id^*[\psi] \\ &= [\phi]\cup[\psi]. \quad // \end{aligned}$$