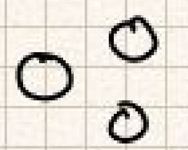
Mflds + Orientability

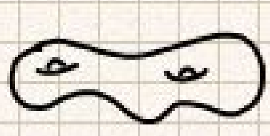
<u>Defn</u> An <u>n-manifold</u> is a space X which is Hausdorff, and locally homeomorphic to $\mathbb{R}^n$.

<u>Rmk</u> i.e, $\forall\, x \in X$, $\exists$ $\mathcal{U} \subset X$ open, $x \in \mathcal{U}$ s.t. $\mathcal{U}$ is homeo to $\mathbb{R}^n$.

<u>Ex</u>

 0-mfld

 1-mfld

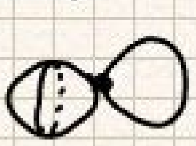
 2-mfld

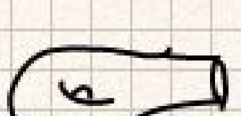
$\mathbb{C}P^n$ 2n-mfld

$\mathbb{R}P^n$, $\mathbb{R}^n$, S^n n-mfld

- Any product of mflds is a mfld.
- Any covering of an n-mfld is an n-mfld.

<u>Non-ex</u>

 $S^n \vee S^m$ n or m $\neq 0$.

 manifold <u>with</u> <u>boundary</u>.

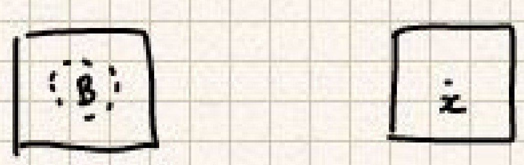
Let X be an n-mfld.

$\forall x \in X$, consider

$$\begin{aligned} H_n(X, X-x) &\xleftarrow{\cong} H_n(U, U-x) && \text{Excision} \\ &\cong H_n(\mathbb{R}^n, \mathbb{R}^n - \{0\}) && U \cong \mathbb{R}^n \\ &\xleftarrow{\cong} H_n(D^n, \partial D^n) \\ &\cong \widetilde{H}_n(S^n) \\ &\cong \mathbb{Z}. \end{aligned}$$

__Defn__ A __local__ __orientation__ o_x of X at x is a choice of generator of
$$H_n(X, X-x).$$

__Rmk__ What __is__ a local orientation?
Given an open ball $x \in B \subset X$,
a choice o_x determines a generator of
$$H_n(X, X-B) \xrightarrow{\cong} H_n(X, X-x).$$
$$o_B \longmapsto o_x$$



You can think of this as a choice of some "color" called a generator on this open ball.

Consider the set $\{(x, o_x)\} = \widetilde{X}$ of pairs (x, o_x) where $x \in X$ and o_x is a local orientation.

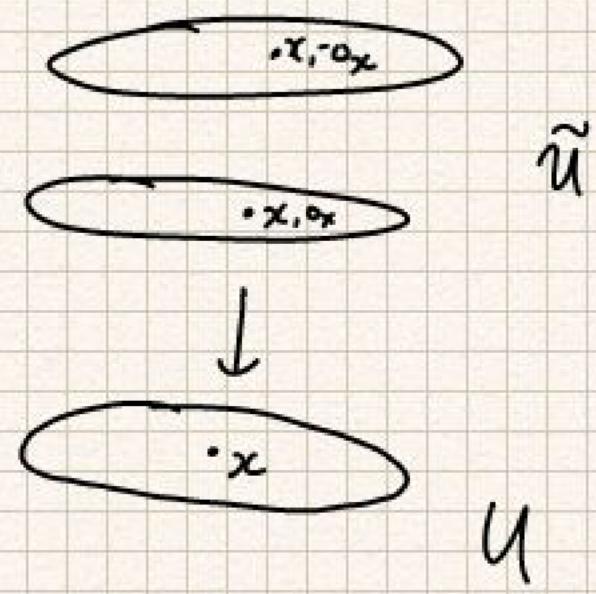
__Defn__ Let $B \subsetneq \mathbb{R}^n \cong U \subset X$ be an open ball,
fix a generator
$$o_B \in H_n(X, X-B) \xleftarrow{\cong} H_n(X, X-U) \cong \mathbb{Z}.$$

Let V_{B, o_B} be the set of (x, o_x) such that

- $x \in B$
- $H_n(X, X-B) \longrightarrow H_n(X, X-x)$ induced by $id: X \longrightarrow X$ sends o_B to o_x.

Prop The collection $\{V_{B,o_B}\}$ is a basis for a topology on $\widetilde{X}$, and the map

$$\widetilde{X} \longrightarrow X$$
$$(x, o_x) \longmapsto x$$

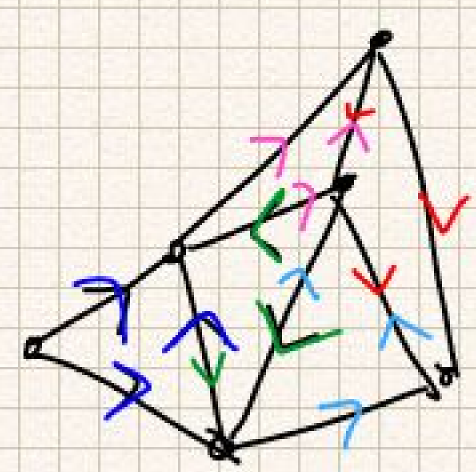


is a 2-sheeted covering, called the orientation cover.

Defn X is called orientable if $\widetilde{X}$ is not connected. An orientation on X is a choice of section of $\widetilde{X} \longrightarrow X$.

Rmk A section is a choice of o_x $\forall$ $x \in X$, all chosen "compatibly": $\forall$ closed curves $\gamma : [0,1] \longrightarrow X$, the lift to $\widetilde{X}$ begins and ends @ the same orientation.

Whether or not a cmpct mfld is orientable has an effect on its top-dimensional homology. Heuristically, if X is triangulated, you can find a non-zero element of $H_n(X)$ by choosing "coherently oriented" simplices that cover all of X. So you'd expect that $H_n(X)$ is non-zero if you can do this.



We won't prove — but we will rely on — the following:

<u>Thm</u> Let X be cmpct, connected n-mfld.

(1) $H_n(X; \mathbb{Z}/2) \cong \mathbb{Z}/2$

"if you ignore signs, coherence is always possible."

(2) X is orientable iff $H_n(X) \cong \mathbb{Z}$

(3) If X is orientable, the map $H_n(X, \phi) \to H_n(X, X - \{x\})$ is an $\cong$ $\forall x \in X$.

so choosing a generator of $H_n(X) \cong H_n(X, \phi) \cong \mathbb{Z}$ fixes a compatible orientation $\forall x \in X$

Rmk If X is a smooth manifold, an orientation is a compatible choice of determinant on each tangent space $T_x X$.

Defn If X is an orientable n-mfld, a choice of generator of $H_n(X) \cong \mathbb{Z}$ is called a fundamental class for X and is written $[X]$.

Examples

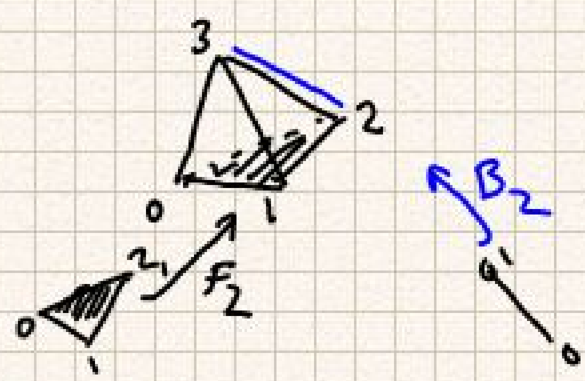
- $\mathbb{R}^n$ is orientable
- In fact, any mfld X w/ $\pi_1(X, x_0) = 0$ $\forall x_0$ is orientable, since orientation cover must be disconnected.
- $\mathbb{RP}^n$, n even, is NOT orientable, since $H_n(\mathbb{RP}^n) = 0$.

Cap product

Defn (Cap product)

$$\frown : C_k \otimes C^\ell \longrightarrow C_{k-\ell}, \quad k \geq \ell$$
$$\sigma \otimes \psi \longmapsto \psi(\sigma \circ F_\ell)\, \sigma \circ B_\ell .$$

"Integrate out ℓ-dimensional component"



Lemma (Basic properties of $\frown$)

(1) $\partial(\sigma \frown \psi) = (-1)^\ell (\partial \sigma \frown \psi - \sigma \frown d\psi)$.

(2) $\frown$ induces a map
$$\frown : H_k \otimes H^\ell \longrightarrow H_{k-\ell} \quad \forall\, \ell \geq k$$

(3) Given $f : X \to Y$, $\alpha \in H_k(X)$, $\psi \in H^\ell(Y)$
$$f_*(\alpha) \frown \psi = f_*(\alpha \frown f^* \psi).$$

$$\begin{tikzcd}
H_k(X) \arrow[r, "f_*"] \arrow[d, "\frown f^*\psi"'] & H_k(Y) \arrow[d, "\frown \psi"] \\
H_{k-\ell}(X) \arrow[r, "f_*"] & H_{k-\ell}(Y)
\end{tikzcd}$$

(4) $\psi(\alpha \frown \phi) = (\phi \smile \psi)(\alpha)$
if $\alpha \in C_{k+\ell}$, $\phi \in C^k$, $\psi \in C^\ell$.

(5)
$$\begin{tikzcd}
H^\ell(X) \arrow[r, "h"] \arrow[d, "\phi \smile"'] & \hom(H_\ell(X), \mathbb{Z}) \arrow[d, "\circ(\phi \frown -)"] \\
H^{k+\ell}(X) \arrow[r, "h"] & \hom(H_{k+\ell}(X), \mathbb{Z})
\end{tikzcd}$$

Thm (Poincaré Duality)

(1) $\forall$ n-dim, cmpct mflds X, $\frown$ induces $\cong$

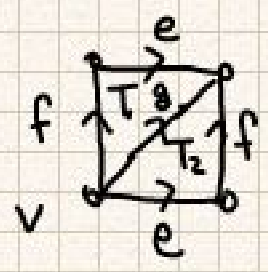
$$H^k(X;\mathbb{Z}/2) \xrightarrow{\sim} H_{n-k}(X;\mathbb{Z}/2)$$

(2) If X is oriented w/ fundamental class $[X]$,

$$[X]\frown : H^k(X) \longrightarrow H_{n-k}(X)$$

is an $\cong$.

Ex $X = T^2$.



T^2 as geom. realization of a ssSet.

$\partial T_1 = e - g + f$

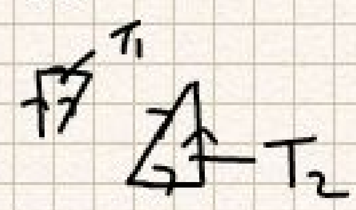
$\partial T_2 = f - g + e$.

$T_1 - T_2$ generates H_2

e, f " H_1 $\qquad e^\vee \in H^1$ so $e^\vee(e) = 1$

v " H_0. $\qquad e^\vee(f) = 0$.

$[T_1 - T_2] \frown e^\vee$



$= e^\vee(T_1|_{01})\, T_1|_{12}$
$\quad - e^\vee(T_2|_{01})\, T_2|_{12}$

$= -1\, T_2|_{12}$

$= -f$

$[T_1 - T_2] \frown f^\vee = e$.

Rmk

$$\begin{tikzcd} H^j(X)\otimes H^k(X) \arrow[d, "\cup"] & H_{n-j}(X)\otimes H_{n-k}(X) \arrow[l, "([X]\frown\ )^{-1}\otimes([X]\frown\ )^{-1}"'] \\ H^{j+k}(X) \arrow[r, "{[X]\frown}"] & H_{n-j-k}(X) \end{tikzcd}$$

This composition defines a product on homology! This is called the intersection product.

Rmk Let M, N be oriented, compact manifolds.

Fix $f: M \longrightarrow X$
$g: N \longrightarrow X$

and let $\sigma = f_*[M]$
$\tau = g_*[N]$.

If f, g are embeddings, can think of $\sigma \frown \tau$ as
$f(M) \cap g(N)$,
counted w/ multiplicity.

Ex $n=3$ $j, k = 1$.

$M \cap N = 2$ circles w/ various orientations. (Homologous to zero in this ex.)

Need orientation on X to assign orientation to $M \cap N$.

Rmk Much more intuitive picture of cup product.

Cor Let X be a compact n-mfld.

(1) $H_k(X;\mathbb{Z}/2\mathbb{Z})$ for $k \le n/2$ determines $H_k(X;\mathbb{Z}/2\mathbb{Z})$ $\forall$ k. $\quad (H_k(X;\mathbb{Z}/2\mathbb{Z}) \cong H_{n-k}(X;\mathbb{Z}/2\mathbb{Z}))$

(2) If X is orientable, and $\mathbb{F}$ is a field, $H_k(X;\mathbb{F})$ for $k \le n/2$ determines $H_k(X;\mathbb{F})$ $\forall$ k. $\quad (H_k(X;\mathbb{F}) \cong H_{n-k}(X;\mathbb{F}))$

(3) If X is orientable and $H_k(X)$ free $\forall$ $k \le n/2$, $H_k(X)$ is determined $\forall$ k. $\quad (H_k(X) \cong H_{n-k}(X)$ if H_k free.)

Pf By univ coeff theorem, we have SES

$$0 \to \mathrm{Ext}^1_R(H_{k-1}(X;R),R) \to H^k(X;R) \to \hom_R(H_k(X;R),R) \to 0.$$

If $H_{k-1}(X;R)$ is a free R-module (always true when R is a field) then

$$H^k(X;R) \cong \hom_R(H_k(X;R),R),$$

since $\mathrm{Ext} = 0$.

By Poincaré duality,

$$H_{n-k}(X;R) \cong H^k(X;R). \quad //$$

Rmk To prove $\hom_R(H_k(X;R),R) \cong H_k(X;R)$, need to know H_k is finitely generated. This is true $\forall$ compact mflds.

Ex

$\mathbb{RP}^6$

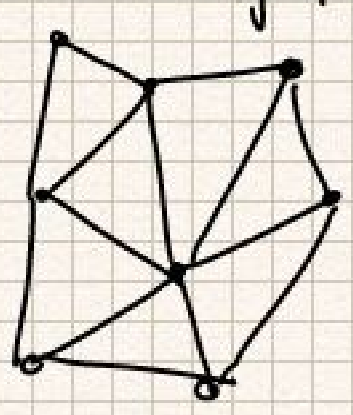
	0	1	2	3	4	5	6
H_*	$\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	0	$\mathbb{Z}/2\mathbb{Z}$	0	$\mathbb{Z}/2\mathbb{Z}$	0
H^*	$\mathbb{Z}$	0	$\mathbb{Z}/2\mathbb{Z}$	0	$\mathbb{Z}/2\mathbb{Z}$	0	$\mathbb{Z}/2\mathbb{Z}$
$H_*(\ ;\mathbb{Z}/2)$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/2$
$H^*(\ ;\mathbb{Z}/2)$	"	"	"	"	"	"	"

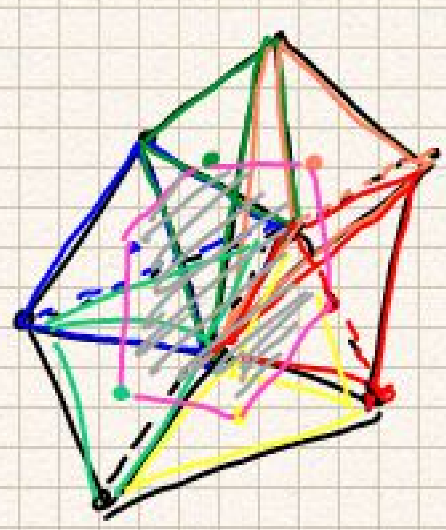
$\mathbb{RP}^5$

	0	1	2	3	4	5
H_*	$\mathbb{Z}$	$\mathbb{Z}/2$	0	$\mathbb{Z}/2$	0	$\mathbb{Z}$
H^*	$\mathbb{Z}$	0	$\mathbb{Z}/2$	0	$\mathbb{Z}/2$	$\mathbb{Z}$

Rmk (Origins of Poincaré Duality)

Given a triangulation of an n-mfld





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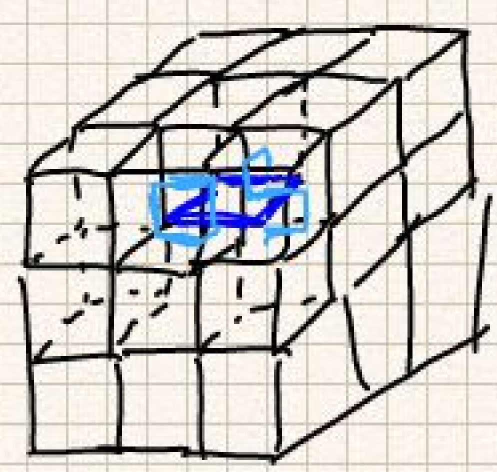

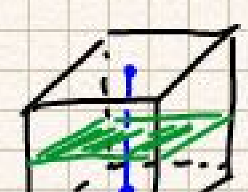

one can construct a "dual" cell structure by replacing every k-simplex w/ a dual $(n-k)$-cell:

τ n-simplex $\longleftrightarrow$ 0-simplex @ center, $\tau^\vee$

σ $(n-1)$-simplex $\longleftrightarrow$ an edge $\sigma^\vee$ from $\tau_0^\vee$ to $\tau_1^\vee$, where $\sigma = \tau_0 \cap \tau_1$.

ρ $(n-2)$-simplex $\longleftrightarrow$ A polygon w/ edges $\sigma_i^\vee$, where $\rho \in \partial\sigma_i$ $\forall i$.

etc. By construction, ∂ maps of original triangulation determine d maps of cochains of dual cell structure.