1. Suppose that $\iota \in \hom(X, X)$ satisfies $\iota \circ f = f \circ \iota = f$ for all $f \in \hom(X, X)$. Then, setting $f = \mathrm{id}_X$, we have $\iota \circ \mathrm{id}_X = \mathrm{id}_X \circ \iota = \mathrm{id}_X$. But, as id_X is the identity, we also have $\mathrm{id}_X \circ \iota = \iota = \iota \circ \mathrm{id}_X$, so $\mathrm{id}_X = \iota$, as desired.

2. Let $\mathcal{C}$ be a category with one object, so $\mathcal{C}_0 = \{*\}$.

   (a) Suppose every morphism of $\mathcal{C}$ is invertible, and let $G = \hom_{\mathcal{C}}(*, *)$.

   By the definition of a category, we know that G is associative, and that G has an identity $e = \mathrm{id}_* \in G$. Furthermore, we are told that for each $f \in G$, there is a $g \in G$ such that $f \circ g = g \circ f = \mathrm{id}_* = e$, so every element of G has an inverse. Therefore, G is a group.

   (b) As above, we know that $\hom_{\mathcal{C}}(*, *)$ is associative with respect to the operation of composition, and that it contains an identity. This is precisely the definition of a monoid.

3. For any group G, let BG be the category with one object, $*$, such that $\hom_{BG}(*, *) = G$, and composition is that of the group G. Define a functor Groups $\to$ Categories such that $G \mapsto BG$.

   If G, G' are two groups, $\hom_{\text{Groups}}(G, G')$ is simply the set of group homomorphisms $G \to G'$. On the other hand, the set $\hom_{\text{Categories}}(BG, BG')$ is the set of all functors $BG \to BG'$, which is equal to the set of all ways to map $hom_{BG}(*, *)$ to $\hom_{BG'}(*, *)$ that send the identity to the identity and preserve composition. But, those are exactly the conditions for which that set map is a group homomorphism, so indeed we have $\hom_{\text{Groups}}(G, G') \cong \hom_{\text{Categories}}(BG, BG')$, as desired.

4. For any R, let R-mod be the category of left R-modules. Consider the assignment $R \mapsto R$-mod, sending a ring to a category.

This clearly defines a map $\text{Rings}_0 \to \text{Categories}_0$, so we now consider what happens to the morphisms. If R, R' are rings and $f \in \hom_{\text{Rings}}(R, R')$, then f gives us a way to regard R' as an R-module by $rr' = f(r)r'$. Thus, we can define $M' = R' \otimes_R M$, which has a natural R'-action and is therefore an R'-module. We will abuse notation a bit here and write $M' = R' \otimes_f M$, to remind us of the effect of the particular homomorphism f we chose. Thus, the map $\hom_{\text{Rings}}(R, R') \to \hom_{\text{Categories}}(R\text{-mod}, R'\text{-mod})$ is given by $f \mapsto (M \mapsto R' \otimes_f M)$.

It is clear that this maps the identity to the identity. However, we see here that it is not strictly a functor, because considering the composition $R \xrightarrow{f} R' \xrightarrow{f'} R''$, we have $R'' \otimes_{f'} (R' \otimes_f M) \cong R'' \otimes_{f' \circ f} M$, however these are not the same module. To address this, we must instead just consider everything up to isomorphism, but note that the problem is technically not correct as stated for this reason.

It then remains to show that $M \mapsto R' \otimes_f M$ actually defines a functor (up to isomorphism) from R-mod to R'-mod. It is clear that it is a well-defined assignment on the level of objects, so consider $\phi \in \hom_{R-\text{mod}}(M, N)$ where M, N are R-modules. The induced map on the R'-modules ϕ' is given by $r' \otimes m \mapsto r' \otimes \phi(m) \in R' \otimes_f N$, given us a bonafide homomorphism of R'-modules. It is clear that this will map the identity to the identity, and will also be well-behaved with respect to composition. Therefore, f indeed defines a functor $R\text{-mod} \to R'\text{-mod}$, so our assignment $R \mapsto R\text{-mod}$ indeed gives us a functor.

5. Let $\mathbb{Z}/2\mathbb{Z}$ be the $\mathbb{Z}$-module with 2 elements, considered as a chain complex concentrated in degree 0.

   (a) Let $A_\bullet$ be the complex given by $A_1 = A_0 = \mathbb{Z}$, $A_i = 0$ if $i \neq 0, 1$, where the map $d_1 : A_1 \to A_0$ is given by multiplication by 2. Consider the chain map $A_\bullet \to \mathbb{Z}/2\mathbb{Z}$ given by $1 \mapsto 1$ in degree 0 (we must have the zero map in all other degrees). Then, for $i \neq 0$, we have $H_i(A_\bullet) \cong 0 \cong H_i(\mathbb{Z}/2\mathbb{Z})$, and $H_i(A_\bullet) \cong \mathbb{Z}/2\mathbb{Z}$, with the generator being sent to the generator, giving us an isomorphism in homology.

(b) Let $A_\bullet$ be the complex given by $A_1 = B_0 = \mathbb{Z}$, $A_i = 0$ if $i \neq 0, 1$, where the map $d_1 : A_1 \to A_0$ is given by multiplication by n. Consider the chain map $A_\bullet \to \mathbb{Z}/n\mathbb{Z}$ given by $1 \mapsto 1$ in degree 0 (and all other maps are again constrained to be the zero map). Then, for $i \neq 0$, we have $H_i(A_\bullet) \cong 0 \cong H_i(\mathbb{Z}/n\mathbb{Z})$, and $H_i(A_\bullet) \cong \mathbb{Z}/n\mathbb{Z}$, with the generator being sent to the generator, giving us an isomorphism in homology.

(c) The only map $\mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}$ is the zero map, and as A_0 is going to be a direct sum of copies of $\mathbb{Z}$, this means that the map $\mathbb{Z}/n\mathbb{Z} \to A_0$ must be the zero map. This is already enough to tell us that this will fail in this direction, as the induced map in H_0 is going to be $\mathbb{Z}/n\mathbb{Z} \to 0$, which is not an isomorphism.

6. For a chain complex $A_\bullet = (\cdots \to A_1 \xrightarrow{d} A_0 \to A_{-1} \to \cdots)$, and an abelian group B, let $\hom^i(A_\bullet, B) = \hom_{\mathrm{Ab}}(A_i, B)$.

Consider the sequence $\hom^\bullet(A_\bullet, B) = (\cdots \to \hom^i(A_\bullet, B) \to \hom^{i+1}(A_\bullet, B) \to \cdots)$ with differential $\delta(f) = f \circ d$.

Suppose that $f \in \hom^n(A, B)$. Then, $\delta^2(f) = \delta(\delta(f)) = (\delta(f)) \circ d = (f \circ d) \circ d = f \circ d^2 = f \circ 0 = 0$, so $\hom^\bullet(A_\bullet, B)$ is indeed a chain complex.

7. Let $A_\bullet$ be as in 5(b), and let $B = \mathbb{Z}/m\mathbb{Z}$. Here, we will make the choice of cohomological grading for the duals.

(a) We have $\hom^0(A, B) \hom^1(A, B) = \hom(\mathbb{Z}, \mathbb{Z}/m\mathbb{Z}) = \mathbb{Z}/m\mathbb{Z}$, and $\hom^i(A, B) = 0$ for all $i \neq 0, 1$. In particular, this means that if $i < 0$, we have that the H^i's are zero as well.

(b) Because $A_\bullet$ is zero in degree 2 and higher, $\hom_\bullet(A, B)$ is as well, and therefore so are the H^i's.

(c) $\delta : \hom^0(A, B) \to \hom^1(A, B)$ is given by multiplication by n, so $H^0 = \ker \delta$, which is equal to the set of all elements of $\mathbb{Z}/m\mathbb{Z}$

annihilated by n. This is a subgroup that is isomorphic to $\mathbb{Z}/g\mathbb{Z}$, where $g = \gcd(m,n)$, so $H^0(\hom^\bullet(A,B)) = \mathbb{Z}/g/ZZ$.

We also know that $\ker(\delta : \hom^1(A,B) \to /hom^2(A,B)) = \hom^1(A,B) = \mathbb{Z}/m\mathbb{Z}$, and $\mathrm{im}(\delta : \hom^0(A,B) \to \hom^1(A,B)) = n\mathbb{Z}/m/ZZ = g\mathbb{Z}/m\mathbb{Z}$. Therefore, we have that $H^1 \cong \mathbb{Z}/g\mathbb{Z}$ as well.

8. We have a clearly defined set of objects and morphisms between them. Since $\hom_{\mathcal{C}^{op}}(X,X) = \hom_{\mathcal{C}}(X,X)$, by definition, id_X is the same in this category, and functions in exactly the same way, so $\mathrm{id}_X \circ f = f \circ \mathrm{id}_X = f$ for any $f \in \hom_{\mathcal{C}}(X,X)$.

   Suppose now that $f \in \hom_{\mathcal{C}^{op}}(W,X), g \in \hom_{\mathcal{C}^{op}}(X,Y), h \in \hom_{\mathcal{C}^{op}}(Y,Z)$. Letting f^*, g^*, h^* be the corresponding morphisms in the original category, we have $f \circ (g \circ h) = (h^* \circ g^*) \circ f^* = h^* \circ (g^* \circ f^*) = (f \circ g) \circ h$, by the associativity of morphisms in $\mathcal{C}$, along with the definition of composition in $\mathcal{C}^{op}$.

   Thus, we can conclude that $\mathcal{C}^{op}$ is a category.

9. Consider the assignment $(\mathcal{C}^{op})_0 \to \text{Sets}$ given by $X \mapsto \hom_{\mathcal{C}}(X,Y) = \hom_{\mathcal{C}^{op}}(Y,X)$. Letting $\mathcal{D} = \mathcal{C}^{op}$, we have an assignment $\mathcal{D} \to \text{Sets}$ given by $X \mapsto \hom_{\mathcal{D}}(Y,X)$. If $X_0, X_1 \in \mathcal{D}_0$, and $f \in \hom_{\mathcal{D}}(X_0,X_1)$, then f is taken to the map $\phi_f : \hom_{\mathcal{D}}(Y,X_0) \to \hom_{\mathcal{D}}(Y,X_1)$ given by post-composition with f ($\phi_f(g) = f \circ g$, for $g \in \hom_{\mathcal{D}}(Y,X_0)$). Setting $f = \mathrm{id}_X$ we have $\phi_{\mathrm{id}_X})(g) = \mathrm{id}_X \circ g = g$, so $\phi_{\mathrm{id}_X} = \mathrm{id}_{\hom_{\mathcal{D}}(Y,X)}$. Therefore, it takes identities to identities.

   Suppose that we also have $f' \in \hom_{\mathcal{D}}(X_1,X_2)$. Then, if $h \in \hom_{\mathcal{D}}(Y,X_0)$, we have $\phi_{f' \circ f}(h) = (f' \circ f) \circ h = f' \circ (f \circ h) = \phi_{f'}(\phi_f(h)) = (\phi_{f'} \circ \phi_f) \circ h$, so associativity holds as well. Therefore, this assignment is indeed a functor.

10. Let $\mathcal{C}$ be the category with one object, and a single morphism (which must be the identity, as noted).

If we choose an object $D \in \mathcal{D}_0$, we can consider the assignment $* \mapsto D$. As long as we map the identity on $*$ to the identity on D, this specifies a functor. Because this is a requirement for a functor, we get exactly one functor from each such assignment.

11. Let F and G be two functors from $\mathcal{C}$ to $\mathcal{D}$, where $\mathcal{C}$ is the category from the previous problem. It is clear that specifying a natural transformation $\eta : F \to G$ gives us a morphism in $\mathcal{D}$ as we specify $\eta_* : F(*) \to G(*)$.

    On the other hand, F, G are both completely determined by the images $F(*), G(*)$. To specify a natural transformation, we need only specify the map $\eta_* : F(*) \to G(*)$ corresponding to the single object $* \in \mathcal{C}$. Because F, G are functors, we must have $F(f) = \mathrm{id}_{F(*)}, G(f) = \mathrm{id}_{G(*)}$, so the diagram automatically commutes, and so we get a natural transformation from any morphism in $\mathcal{D}$.

12. Let $\mathcal{C}$ be the category with two objects, X_0 and X_1, such that $\hom_{\mathcal{C}}(X_0, X_1)$ has a single element called f, $\hom_{\mathcal{C}}(X_1, X_0)$ is empty, and $hom_{\mathcal{C}}(X_0, X_0)$ and $\hom_{\mathcal{C}}(X_1, X_1)$ have a single element (the identity).

    Fix two functors $F, G : \mathcal{C} \to \mathcal{D}$.

    The only nontrivial condition to check for $\eta : F \to G$ to be a natural transformation is that arising from f. This is the commutative square

    $$\begin{array}{ccc} F(X_0) & \xrightarrow{F(f)} & F(X_1) \\ \downarrow & & \downarrow \\ G(X_0) & \xrightarrow{G(f)} & G(X_1) \end{array}.$$

    Likewise, any commutative square in $\mathcal{D}$ for which two of the morphisms are specified by F and G can be written in this manner, and therefore determines a natural transformation $F \to G$.

13. Let $\mathcal{C}$ be the category of commutative rings (because I am dyslexic) and $\mathcal{D}$ the category of groups. Let $F : \mathcal{C} \to \mathcal{D}$ assign to each ring its

group of units, and let $G : \mathcal{C} \to \mathcal{D}$ assign to each ring R the group $SL_n(R)$.

(a) Let $f \in \hom_{\mathcal{C}}(R, R')$. Then, $F(f)$ is the restriction of f to the set of units of R. On the other hand, $G(f)$ is maps $n \times n$ matrices with entries in R to $n \times n$ matrices with entries in R', and in particular, maps invertible matrices with entries in R to invertible matrices with entries in R'.

(b) Define $\eta_R : G \to F$ by the determinant. Then, for any rings R, R', and a morphism $f : R \to R'$, we have the diagram $\begin{array}{ccc} G(R) & \xrightarrow{\eta_R} & F(R) \\ \downarrow & & \downarrow \\ G(R') & \xrightarrow{\eta_{R'}} & F(R') \end{array}$

Let $A \in G(R) = GL_n(R)$. Then, $F(f)(\det(A)) = f(\det A)$, as $\det A$ is a unit of R, and is therefore in R. On the other hand, $\det(G(f)(A)) = \det(f(A)) = f(\det A)$, by the multiplicative nature of ring homomorphisms. Thus, we see that the diagram commutes, and therefore η is a natural transformation.

14. Fix two categories $\mathcal{C}, \mathcal{D}$, and define a new category $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ whose objects are functors from $\mathcal{C}$ to $\mathcal{D}$, and whose morphisms are the natural transformations between them, with composition defined in the natural way.

(a) Suppose that $\eta : F \to G, \omega : G \to H$ are natural transformations, and consider $\omega \circ \eta$. If we draw the diagram for this for any two objects $X, Y \in \mathcal{C}$ and any morphism $f \in \hom_{\mathcal{C}}(X, Y)$, then by the definition of $\omega \circ \eta$, we can factor it into the diagrams for ω and η, both of which commute. Therefore, the total diagram commutes as well, so $\omega \circ \eta$ is indeed a natural transformation.

(b) The identity morphism id_F is the natural transformation η for which $\eta_X = \mathrm{id}_{F(X)}$ for each object $X \in \mathcal{C}_0$.

15. Using the idea from (9), the functor $\mathcal{C} \to \text{Fun}(\mathcal{C}^{op}, \text{Sets})$ is given by sending each $X \in \mathcal{C}_0$ to $\hom_{\mathcal{C}}(-, X) = \hom_{\mathcal{C}^{op}}(X, -)$.

    This clearly gives a well-defined map on classes that is covariant on morphisms. We see that id_X maps to the natural transformation η for which $\eta_Y = \text{id}_{F(Y)} = \text{id}_{\hom(Y,X)}$ for each $Y \in \mathcal{C}$. From the previous exercise, we see that natural transformations compose in the correct way as well, so this assignment behaves correctly on morphisms. From this, we conclude that it is indeed a functor.