__Exercises__. (Due Fri, Sep 13th, 2013)

Recall a category $\mathcal{C}$ is the data

- $\mathcal{C}_0$ objects of $\mathcal{C}$
- $\forall\ X, Y \in \mathcal{C}_0$, a set
$$\hom_{\mathcal{C}}(X,Y)$$
- a rule/product/composition law
$$\hom_{\mathcal{C}}(X,Y) \times \hom_{\mathcal{C}}(Y,Z) \longrightarrow \hom_{\mathcal{C}}(X,Z)$$
for any triplet $X, Y, Z \in \mathcal{C}_0$, which satisfies __associativity__:

$$\begin{tikzcd}
 & \overset{(f,\ g,\ h)}{\hom(W,X)\times\hom(X,Y)\times\hom(Y,Z)} \arrow[dl] \arrow[dr] & \\
(f, h\circ g) \in \hom(W,X)\times\hom(X,Z) \arrow[dr] & & \overset{(g\circ f,\ h)}{\rotatebox{90}{$\in$}}\ \hom(W,Y)\times\hom(Y,Z) \arrow[dl] \\
 & \hom(W,Z) &
\end{tikzcd}$$

$$(h\circ g)\circ f = h\circ(g\circ f).$$

- $\forall\ X \in \mathcal{C}_0$, an element in $\hom_{\mathcal{C}}(X,X)$ called the __identity__ of X, written id_X, s.t. $\forall\ Y$, and $\forall$ $f \in \hom_{\mathcal{C}}(X,Y)$, $g \in \hom_{\mathcal{C}}(Y,X)$,
$$f\circ id_X = f, \qquad id_X\circ g = g.$$

Recall also that a <u>functor</u> from $\mathcal{C}$ to $\mathcal{D}$ is the data

- an assignment $F(X) \in \mathcal{D}_0$ for every object $X \in \mathcal{C}_0$

- For every morphism $f: X \to Y$ in $\mathcal{C}$, a morphism

$$F(f): F(X) \to F(Y) \quad \text{in } \mathcal{D},$$

such that

and $F(\mathrm{id}_X) = \mathrm{id}_{F(X)}$

$$\underbrace{F(g \circ f)}_{\text{composition in } \mathcal{C}} = \underbrace{F(g) \circ F(f)}_{\text{composition in } \mathcal{D}}$$

A functor is a morphism in the category of categories.

(1) Show that if an element $\tau \in \hom(X, X)$ satisfies
$$\tau \circ g = g, \quad f \circ \tau = f \quad \forall \quad f \in \hom(X, Y),\ g \in \hom(Y, X)$$
then $\tau = id_X$.

(2) Let $\mathcal{C}$ be a category with one object, so $\mathcal{C}_0 = \{*\}$.

(a) Suppose every morphism of $\mathcal{C}$ is invertible. Show $\hom_{\mathcal{C}}(*, *)$ is a group.

(b) More generally, show $\hom_{\mathcal{C}}(*, *)$ is a monoid (we've removed the invertibility assumption).

(3) For any group G, let BG be the category with one object, $*$, s.t. $\hom_{BG}(*, *) = G$, and composition is that of the group G. Define a functor

$$\text{Groups} \longrightarrow \text{Categories}$$

such that $G \mapsto BG$, satisfying the property that

$$\hom_{\text{Groups}}(G, G') \cong \hom_{\text{Categories}}(BG, BG')$$

for any two groups G and G'.

(4) For any ring R, let R-mod be the category of left R-modules. Show that the assignment $R \mapsto R\text{-mod}$ defines a functor

$$\text{Rings} \longrightarrow \text{Categories}.$$ (You define its effect on morphisms.

(5) Let $\mathbb{Z}/2\mathbb{Z}$ be the $\mathbb{Z}$-module with 2 elements, considered as a chain complex concentrated in degree 0:

$$\cdots \xrightarrow{d} 0 \xrightarrow{d} 0 \xrightarrow{d} \mathbb{Z}/2\mathbb{Z} \xrightarrow{d} 0 \xrightarrow{d} 0 \longrightarrow \cdots$$

degree: $\quad 2 \quad 1 \quad 0 \quad -1 \quad -2$

(a) Construct a chain complex $A_\bullet$ such that

- there is a map of chain complexes $A_\bullet \longrightarrow \mathbb{Z}/2\mathbb{Z}$ inducing an isomorphism on homology
- A_i is a free abelian group for all i.
  (i.e., $A_i \cong \mathbb{Z}^{n_i}$ for some $n_i \in \{0,1,2,\cdots\}$.)

(b) Do the same for $\mathbb{Z}/n\mathbb{Z}$, rather than $\mathbb{Z}/2\mathbb{Z}$. ($n \geq 2$).

(c) Show there does NOT exist a map going the other way — ie, a map $g: \mathbb{Z}/n\mathbb{Z} \longrightarrow A_\bullet$ — such that g induces the inverse to f in homology.

(6) For a chain complex $A_\bullet = (\cdots \longrightarrow A_1 \xrightarrow{d} A_0 \longrightarrow A_{-1} \longrightarrow \cdots)$, and an abelian group B, let

$$\hom_i(A_\bullet, B) = \hom_{\text{Abelian Groups}}(A_i, B), \qquad \left(\begin{array}{c}\text{sometimes written}\\ \hom_i(A,B)\end{array}\right)$$

and show $\hom_i$ defines a chain complex

$$\left(\cdots \longrightarrow \hom_i(A_\bullet, B) \longrightarrow \hom_{i-1}(A_\bullet, B) \longrightarrow \cdots\right) =: \hom_\bullet(A, B)$$

i+1

with differential $\delta(f) = f \circ d$. (i.e., show $\delta^2 = 0$).

(7) Let $A_\bullet$ be your solution to (5)(b), so $A_\bullet$ and $\mathbb{Z}/n\mathbb{Z}$ have isomorphic homology. ($H_i(A_\bullet) = \mathbb{Z}/n\mathbb{Z}$ when $i=0$, and 0 otherwise). Let $B = \mathbb{Z}/m\mathbb{Z}$.

(a) Show that all the homology groups of $\hom_\bullet(A, B)$ are concentrated in non-positive degree. (i.e., $H_i = 0$ wherever $i > 0$).

(b) Further show that $H_i = 0$ wherever $i \leq -2$.

(c) Compute $H_0(\hom_\bullet(A, B))$ and $H_{-1}(\hom_\bullet(A, B))$.

(d)* Suppose a classmate had a different (and correct) chain complex A' as a solution to (5)(b). Show your answers to 7(c) would be the same, and that the properties of 7(a) and 7(b) hold.

You have now computed Ext^i of $\mathbb{Z}/n\mathbb{Z}$ and $\mathbb{Z}/m\mathbb{Z}$, since

$$\mathrm{Ext}^i(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}/m\mathbb{Z}) \cong H_{-i}(\hom(A, B)).$$

(8) Let $\mathcal{C}$ be a category. We define a new category $\mathcal{C}^{op}$ as follows:

- $(\mathcal{C}^{op})_0 := \mathcal{C}_0$

- $\forall\ X, Y \in (\mathcal{C}^{op})_0$,
$$\hom_{\mathcal{C}^{op}}(X,Y) = \hom_{\mathcal{C}}(Y,X)$$

- $\forall\ X, Y, Z \in (\mathcal{C}^{op})_0$,

$$\hom_{\mathcal{C}^{op}}(X,Y) \times \hom_{\mathcal{C}^{op}}(Y,Z) \longrightarrow \hom_{\mathcal{C}^{op}}(X,Z)$$

is induced by the composition law of $\mathcal{C}$:

$$\begin{tikzcd}
\hom_{\mathcal{C}}(Y,X) \times \hom_{\mathcal{C}}(Z,Y) \arrow[d, "\cong"'] \arrow[dr, dashed, "\text{composition for } \mathcal{C}^{op}"] & \\
\hom_{\mathcal{C}}(Z,Y) \times \hom_{\mathcal{C}}(Y,X) \arrow[r] & \hom_{\mathcal{C}}(Z,X).
\end{tikzcd}$$

Show $\mathcal{C}^{op}$ is a category.

(9) Let $\mathcal{C}$ be a category. A functor $\mathcal{C}^{op} \to \mathcal{D}$ is called a <u>contravariant</u> functor from $\mathcal{C}$ to $\mathcal{D}$. Fix an object Y. Consider an assignment $(\mathcal{C}^{op})_0 \to \text{Sets}$ which sends X to $\hom_{\mathcal{C}}(X,Y)$. Show this defines a contravariant functor
$$\mathcal{C}^{op} \longrightarrow \text{Sets}.$$
(You must define the morphisms.)

(10) Let $\mathcal{C}$ be the category with one object, and a single morphism. (Note this must be the identity.) Show that a functor from $\mathcal{C}$ to $\mathcal{D}$ is the same thing as choosing an object of $\mathcal{D}$.

(11) Let F and G be two functors from $\mathcal{C}$ to $\mathcal{D}$. A <u>natural</u> <u>transformation</u> η from F to G is the data of

note η_x is a morphism in $\mathcal{D}$

$$\text{a map } \eta_X : F(X) \to G(X)$$

for all objects $X \in \mathcal{C}_0$, such that for all morphisms $f: X \to Y$ in $\mathcal{C}$, the diagram

$$\begin{tikzcd} F(X) \arrow[r, "F(f)"] \arrow[d, "\eta_X"'] & F(Y) \arrow[d, "\eta_Y"] \\ G(X) \arrow[r, "G(f)"'] & G(Y) \end{tikzcd}$$

commutes.

If $\mathcal{C}$ is the category from (10), show that a natural transformation from F to G is the same thing as a morphism in $\mathcal{D}$.

(12) Let $\mathcal{C}$ be the category with two objects, X_0 and X_1, such that
- $\hom_{\mathcal{C}}(X_0, X_1)$ has a single element called f,
- $\hom_{\mathcal{C}}(X_1, X_0)$ is empty, and
- $\hom_{\mathcal{C}}(X_0, X_0)$ and $\hom_{\mathcal{C}}(X_1, X_1)$ have a single element (called the identity).

Fix two functors F, G from $\mathcal{C}$ to $\mathcal{D}$. ($\mathcal{D}$ is any category)

Show that a natural transformation from F to G is the choice of a commutative square in $\mathcal{D}$, where two of the morphisms are specified by F and by G.

(13) Let $\mathcal{C}$ = Rings, and $\mathcal{D}$ = Groups. Consider

- $F: \mathcal{C} \to \mathcal{D}$ a functor assigning to each ring R the group of units of the ring. (ie, $F(R)$ is the set of elements $r \in R$ s.t. $\exists\, r^{-1} \in R$ w/ $r^{-1} r = 1$, $r \cdot r^{-1} = 1$, w/ group multiplication inherited from R.)

- $G: \mathcal{C} \to \mathcal{D}$ a functor assigning to each R the intersection of the center of R with the units of R.

(a) ~~Show~~ ~~Define~~ Understand the effects of F and G on the morphisms of $\mathcal{C}$ if I tell you "$F(f)$ and $G(f)$ are induced by the map $f: R \to R'$."

(b) Show that the inclusion ~~$F(R)$~~ $G(R) \xrightarrow{\eta_R} F(R)$ defines a natural transformation from G to F.

(14) ~~(13)~~ ~~Show that~~ Fix two categories $\mathcal{C}$ and $\mathcal{D}$.

We define a new category $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ as follows:

- $(\mathrm{Fun}(\mathcal{C}, \mathcal{D}))_0$ is the set of all functors from $\mathcal{C}$ to $\mathcal{D}$

- If F, G are two functors, then

$$\hom_{\mathrm{Fun}(\mathcal{C},\mathcal{D})}(F, G)$$

is the set of natural transformations from F to G.

- Given two natural transformations η and ω, ~~their composition is~~ where

  η is a nat. trans. from F to G, and
  ω is a nat. trans. from G to H, composition

is defined by taking

$$\cancel{(\eta \circ \omega)_x :=} \quad (\omega \circ \eta)_x := \omega_x \circ \eta_x$$

i.e.,

$$\begin{tikzcd} F(x) \arrow[r, "\eta_x"] \arrow[rr, bend right, "(\omega\circ\eta)_x"'] & G(X) \arrow[r, "\omega_x"] & H(X) \end{tikzcd}.$$

(a) ~~Prove~~ Verify that ~~composition is~~ $(\omega \circ \eta)$ is a natural transformation.

(b) What ~~is~~ is the identity morphism for a functor F?

(15) Show that the idea from (9) defines a functor

$$\mathcal{C} \longrightarrow \mathrm{Fun}(\mathcal{C}^{op}, \mathrm{Sets}).$$

This is called the *Yoneda embedding*.