Homework Two

This week, we'll be proving (or stating) the following:

(1) Let $X_\bullet$ be a semisimplicial set such that
$$|X_\bullet| \cong X$$
for some space X. Then

$$H_n(Ch(X_\bullet)) \cong H_n(X) \quad \forall\, n \in \mathbb{Z}$$

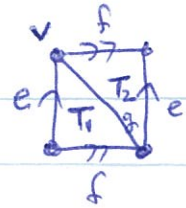
singular homology of X

(2) If $f, g : X \to Y$
are homotopic, then they induce the same map on homology.

(In particular, if X and Y are homotopy equivalent, they have $\cong$ homology groups.)

Hand in problems 1, 2, 3, 6, 8, 9, 10, 12, 15, 16.

Due Friday, Sept. 20th

(1) Let  be a semisimplicial model for the torus, $T = S^1 \times S^1 =$  .

Specifically, let

$$X_2 = \{T_1, T_2\} \qquad X_n = \phi, \ n \geq 3$$

$$X_1 = \{e, f, g\}$$

$$X_0 = \{v\}$$

w/ face maps

$$\begin{aligned} d_0 T_1 &= g & \qquad d_0 T_2 &= e \\ d_1 T_1 &= f & d_1 T_2 &= f \\ d_2 T_1 &= e & d_2 T_2 &= g. \end{aligned}$$

(a) Compute $H_*(T)$. (Convince yourself that $|X_\bullet| \cong$ torus)

(b) Let $Y_\bullet$ be semisimplicial set where $Y_1 = \{a, b\}$, $Y_0 = \{u\}$, $Y_n = \phi$ $n \geq 2$.

~~Compute~~ Consider the maps

$$\begin{aligned} \psi: Y_\bullet &\longrightarrow X_\bullet & \qquad \text{and} \qquad \phi: Y_\bullet &\longrightarrow X_\bullet \\ a &\longmapsto e & a &\longmapsto e \\ b &\longmapsto f & b &\longmapsto g. \end{aligned}$$

There are maps $\infty \overset{\psi}{\underset{\phi}{\rightrightarrows}}$  . Draw them, clearly.

(c) Compute the kernel and cokernel of the maps ψ_*, ϕ_* on H_0, H_1, and H_2.

(d) Prove ψ and ϕ are not homotopic.

(2) Let A and B be topological spaces. ~~and~~
Prove

$$H_n(A \amalg B) \cong H_n(A) \oplus H_n(B) \qquad \forall\, n \in \mathbb{Z}.$$

disjoint union.

(3) Let $Y_\bullet$ be a semisimplicial set where

$$Y_1 = \text{a finite set w/ cardinality } N \geq 1$$
$$Y_0 = \{*\}$$
$$Y_n = \emptyset \quad \forall\, n \geq 2.$$

(a) Draw $|Y_\bullet|$, indicate its dependence on N.

(b) Compute the homology of $|Y_\bullet|$.

(4) Let S^n be the n-sphere, so

$$S^n := \{\vec{x} \in \mathbb{R}^{n+1} \mid \|\vec{x}\| = 1\}.$$

(a) Show $\mathbb{R}^{n+1} \setminus \{\vec{0}\}$ is homotopy equivalent to S^n.

(b) Let $k \geq 1$, and fix k distinct $\underbrace{\text{lines}}_{L_1, \dots, L_k}$ in $\mathbb{R}^3$, all passing through the origin.
Compute the homology of

$$\mathbb{R}^3 \setminus \{L_1 \cup \cdots \cup L_k\}.$$

(5) (a) Let L_z be the z-axis of $\mathbb{R}^3$, and C the unit circle on the xy-plane.

Compute $H_*(\mathbb{R}^3 \setminus (L_z \cup C))$.

(b) * Compute $H_*(\mathbb{R}^3 \setminus C)$.

(6) Recall that in Δ_{inj}, for $i<j$, $\delta^i \delta^{j-1} = \delta^j \delta^i$.

Note this implies that for any semisimplicial set $X_\bullet$,

$$d_{j-1} d_i = d_i d_j \quad \text{wherever } i<j.$$

i.e.,

$$\begin{tikzcd} X_{n+1} \arrow[r, "d_i"] \arrow[d, "d_j"'] \arrow[dr, phantom, "\circlearrowleft"] & X_n \arrow[d, "d_{j-1}"] \\ X_n \arrow[r, "d_i"'] & X_{n-1} \end{tikzcd}$$

commutes.

Prove that if $\partial_n := \sum_{i=0}^{n} (-1)^i d_i$, then $\partial^2 = 0$ for $Ch(X_\bullet)$.

(7) Compute the homology of S^n by

(a) induction on n, using Mayer-Vietoris for the open cover

$$S^n = \mathbb{R}^n \underset{S^{n-1}\times\mathbb{R}}{\cup} \mathbb{R}^n.$$

(b) CW homology

(c)* simplicial homology, by finding a semisimplicial model for S^n.

(8) Recall that for any topological space X, one obtains a semisimplicial set

$$\mathrm{Sing}(X)_\bullet := \mathrm{Maps}(\Delta^0, X) \leftleftarrows \mathrm{Maps}(\Delta^1, X) \Lleftarrow \mathrm{Maps}(\Delta^2, X).$$

Show that a continuous map $X \to Y$ induces a map of semisimplicial sets

$$\mathrm{Sing}(X)_\bullet \longrightarrow \mathrm{Sing}(Y)_\bullet.$$

(We define a map of semisimplical sets $A, B : \Delta_{inj}^{op} \to \mathrm{Sets}$ to be a natural transformation from A to B).

(9) Let $\mathbb{Z}: \text{Sets} \longrightarrow \text{Abelian Groups}$
be the functor that sends a set X to the free Abelian group $\mathbb{Z}X$ generated by X. ie,

$$\mathbb{Z}X := \left\{ \sum_{x_i \in X} a_i x_i \;,\; a_i \in \mathbb{Z},\; x_i \in X \;\middle|\; a_i = 0 \text{ for all but finitely many } i \right\}$$

Show a map of sets $X \longrightarrow Y$ induces a map of abelian groups $\mathbb{Z}X \longrightarrow \mathbb{Z}Y$, and show that $\mathbb{Z}$ is indeed a functor.

(10) Combining (6), (8), and (9), prove that "singular chain complex" defines a functor

$$\begin{aligned} \text{Spaces} &\longrightarrow \text{Chain Complexes} \\ X &\longmapsto C_\cdot(X) \end{aligned}$$

(11) $\forall\, i \in \mathbb{Z}$, show that "the i^{th} homology group" defines a functor

$$\begin{aligned} H_i : \text{Chain Complexes} &\longrightarrow \text{Abelian group} \\ A_\bullet &\longmapsto H_i(A_\bullet). \end{aligned}$$

(12) (a) Prove that $\mathbb{R}^n$ is homotopy equivalent to a point.

(b) Compute/state the homology grps of $\mathbb{R}^n$.

(13) Let $|Y_\bullet|$ be the space from (3).

(a) Draw a picture showing that $|Y_\bullet|$ is homotopy equivalent to $\mathbb{C} \setminus \{N \text{ distinct points}\}$, where the N here is the same N as in (3).

(b) Compute H_* of $\mathbb{C} \setminus \{N \text{ distinct points}\}$.

(c) Draw ~~all~~ ~~some~~ ~~generators~~ a set of curves on $\mathbb{C} \setminus \{N \text{ distinct points}\}$ that generate H_1 of $\mathbb{C} \setminus \{N \text{ distinct points}\}$

(14) Ignoring set-theoretic issues, show that

$$X \sim Y \iff X \text{ is homotopy equivalent to } Y$$

defines an equivalence relation on the "set" of topological spaces.

(P.S. — an equivalence class of $\sim$ is called a "homotopy type".)

(15) Recall that a subspace $A \subset X$ is said to be a <u>retract</u> of X if $\exists$ a map

$$r : X \to X$$

such that $r^2 = r$, and $\text{image}(r) = A$.

(a) Show that if A is a retract of X, then the map

$$A \hookrightarrow X$$

is an injection on homology.

(b) To contrast, provide an example of an inclusion of spaces $B \subset Y$ where the induced map on homology is NOT an injection.

(c) Let $D^2 = \{ z \in \mathbb{C} \mid |z|^2 \leq 1 \}$ and $S^1 = \{ z \mid |z| = 1 \}$. Show that S^1 is not a retract of D^2.

(d) Prove that every continuous map $f : D^2 \to D^2$ must have at least one point for which $f(z) = z$. You have proven the <u>Brouwer</u> <u>fixed</u> <u>point</u> <u>theorem</u>.

(16) In this problem, you will prove that at any given time, there exists at least one pair of antipodal points on our planet for which their temperature and their height above sea level agree. (i.e., $Temp(\vec{x}) = Temp(-\vec{x})$, $Height(\vec{x}) = Height(-\vec{x})$.)

(a) Assume $\exists$ a map $f: S^2 \longrightarrow \mathbb{R}^2$ s.t. $f(x) \neq f(-x)$ $\forall$ $x \in S^2 \subset \mathbb{R}^3$. (So $-x$ is the point antipodal to x.) Show that

$$h(x) = \frac{f(x) - f(-x)}{\| f(x) - f(-x) \|}$$

defines a map $S^2 \longrightarrow S^1$.

(b) * (Very hard.) Prove that if a continuous map $q: S^1 \longrightarrow S^1$ satisfies $q(x) = -q(-x)$, then q is an injection on homology.

(c) Assuming (b), prove the following:

$\forall$ continuous $f: S^2 \longrightarrow \mathbb{R}^2$, $\exists$ $x \in S^2$ s.t. $f(x) = f(-x)$.

This is called the ~~Borsuk~~ Borsuk-Ulam theorem.