

Homework 3: Relative homology and excision

0. Pre-requisites.

The main theorem you'll have to assume is the excision theorem, but only for Problem 6. Recall what this says: Let $A \subset V \subset X$, where the interior of V contains the closure of A . Note there are inclusion of spaces $V - A \rightarrow V$, and $X - A \rightarrow X$. (As usual, the notation $X - A$ means the complement of A inside X .)

THEOREM 0.1 (Excision). The induced map on homology

$$H_n(X - A, V - A) \rightarrow H_n(X, V)$$

is an isomorphism for all n .

0.1. Goals. The goal of these problems is to prove the following fact:

THEOREM 0.2. Let $A \subset X$ be a closed subset such that A is a deformation retract of some open set $V \subset X$. Then there is an isomorphism

$$H_n(X, A) \cong H_n(X/A, pt).$$

In other words, the algebraic gadget called "relative homology" has a very natural geometric interpretation if A is a nice enough closed subset—it looks exactly like the homology of the quotient X/A .

1. Long Exact Sequences of Homology Groups

This is a problem purely about algebra; there is no topology here. Part (c) is tedious, but builds a foundation for the rest of the problems. Let A, B, C be chain complexes, and let $f : A \rightarrow B$ and $g : B \rightarrow C$ be chain maps. Assume that for every integer $n \in \mathbb{Z}$, the sequence of abelian groups

$$0 \longrightarrow A_n \xrightarrow{f_n} B_n \xrightarrow{g_n} C_n \longrightarrow 0$$

is exact.

(a) Prove the following (a sentence should suffice for each step):

(i) For any $c \in C_n$, there is a $b \in B_n$ such that $g_n(b) = c$.

(ii) If c is closed, then $\partial_n^B b$ (the same b from above) is in the kernel of g_{n-1} .

(iii) If c is closed, there is an element $a \in A_{n-1}$ such that $f_{n-1}(a) = \partial_n^B b$.

For another proof of the above result, trying Googling Jill Clayburgh, It's My Turn, Snake Lemma Proof.

(b) Prove that the map $\partial_n : H_n(C) \rightarrow H_{n-1}(A)$ given by $\partial([c]) := [a]$ is a well-defined map on homology. Note this ∂ is *not* meant to be the differential of a chain complex; unfortunately, the common notation for this map is ∂ , and I chose a poor convention of writing the chain complex differential by the same letter.

(c) Note that we now have maps

$$\cdots \xrightarrow{\partial_{n+1}} H_n(A) \xrightarrow{f_*} H_n(B) \xrightarrow{g_*} H_n(C) \xrightarrow{\partial_n} H_{n-1}(A) \xrightarrow{f_*} \cdots$$

where the ∂ are the maps from the previous part of this problem. Prove the following. (These will take more than just a sentence.)

(i) $\text{image}(f_*) = \ker(g_*)$.

(ii) $\text{image}(g_*) = \ker(\partial_n)$.

(iii) $\text{image}(\partial_n) = \ker(f_* : H_{n-1}(A) \rightarrow H_{n-1}(B))$.

You have just proven the following: **Theorem.** Whenever f and g are chain maps defining short exact sequences as above, we have a long exact sequence in homology groups.

Even if you could not prove everything in part (c), the only thing you will need in the following problems is the construction of the map ∂ . So make sure you understand that. You may invoke this theorem throughout the rest of the problems.

2. Homology of a pair, I

Let X be a space, and $A \subset X$ a subspace. Here is an algebraic analogue of X/A .

- (a) Show that the map $j_n : C_n(A) \rightarrow C_n(X)$ induced by the inclusion $A \rightarrow X$ is an injection of abelian groups.
- (b) For each n , define an abelian group $C_n(X, A)$ by the quotient $C_n(X)/j_n(C_n(A))$. Show that the sequence of groups $C_n(X, A)$ forms a chain complex, where the differential for this chain complex, denoted d_n , is induced by that of X . We will denote its homology by the notation $H_n(X, A)$.

The homology $H_\bullet(X, A)$ is called *the homology of the pair* (X, A) .

3. Homology of a pair, II

- (a) Let α be an element of $C_n(X)$. Assume that $\partial\alpha$ is in the image of $j_n(C_n(A))$. Show that α defines a homology class in $H_n(X, A)$. (The geometric interpretation is that $H_n(X, A)$ captures the shapes in X whose boundaries are fully contained in A .)
- (b) Show (for instance, by citing the previous problems) that there is a long exact sequence of homology groups

$$\dots \rightarrow H_n(A) \rightarrow H_n(X) \rightarrow H_n(X, A) \rightarrow H_{n-1}(A) \rightarrow \dots \rightarrow H_0(X, A) \rightarrow 0.$$

4. Easy pairs

- (a) If A is empty, prove that $H_n(X) \cong H_n(X, A)$. (Note that by definition, the free abelian group on the empty set is the 0 group.) Take care: The lefthand side is the singular homology of X , as we've defined in class. The righthand side is the relative homology, as defined in the problems of this exam.
- (b) Let X be non-empty and let A be a point inside X . Show that $H_n(X, A)$ is isomorphic to $H_n(X)$ whenever $n \geq 1$. If $H_0(X) \cong \mathbb{Z}^N$, show that $H_0(X, A) \cong \mathbb{Z}^{N-1}$. (Note that in particular, if X is path-connected, $H_0(X, A)$ is the zero group.)
- (c) Fix some space V and let $A \subset V$. Assume that the inclusion $i : A \rightarrow V$ is a homotopy equivalence. Show that $H_n(V, A) \cong 0$ for all n .

5. Homology of a pair, III

- (a) Let $A' \subset X'$. A *map of pairs* $f : (X, A) \rightarrow (X', A')$ is a continuous map $f : X \rightarrow X'$ such that $f(A) \subset A'$. Show that there is a category **Pairs** whose objects are pairs (X, A) with $A \subset X$, and morphisms are maps of pairs, and composition is induced by the usual composition for continuous maps.
- (b) Show that a map of pairs induces a map of chain complexes $C_n(X, A) \rightarrow C_n(X', A')$; show that this defines a functor $\mathbf{Pairs} \rightarrow \mathbf{Chain}$ to the category of chain complexes.
- (c) Show (for instance, by citing homework) that for every n , we have a functor from **Pairs** to **Abelian Groups** which sends a map f to the map

$$f_* : H_n(X, A) \rightarrow H_n(X', A').$$

So this problem shows that relative homology can be thought of as an invariant of a pair of spaces (X, A) ; i.e., of how A sits inside X .

6. An exercise in excision

Let $A \subset V \subset X$ be subspaces such that A is a closed subset of X , and V is an open subset of X .

- (a) Show that there is a commutative diagram of pairs of spaces

$$\begin{array}{ccc} (X - A, V - A) & \xrightarrow{j} & (X, V) \\ q|_{X-A} \downarrow & & \downarrow q \\ (X/A - A/A, V/A - A/A) & \xrightarrow{j'} & (X/A, V/A). \end{array}$$

That is, show that $q \circ j = j' \circ q|_{X-A}$ as maps of spaces. Here $q : X \rightarrow X/A$ is the quotient map.

- (b) Show that the restriction of this quotient map to the space $X - A$ and to $V - A$ induces homeomorphisms

$$q|_{X-A} : (X - A) \rightarrow (X/A - A/A) \quad \text{and} \quad q|_{V-A} : (V - A) \rightarrow (V/A - A/A).$$

- (c) Show that q induces an isomorphism on relative homology,

$$q_* : H_n(X, V) \cong H_n(X/A, V/A) \quad \text{for all } n.$$

- (d) Where did you use that A is a closed subset of X ?

Note that we are almost done with the proof that $H_\bullet(X, A) \cong H_\bullet(X/A, pt)$. We just need to replace V with A in the isomorphism from part (c). We will show we can do this when V retracts to A .

7. Long exact sequence of a triple

- (a) Let $A \subset V \subset X$. Show that there are chain maps

$$f : C_\bullet(V, A) \rightarrow C_\bullet(X, A), \quad g : C_\bullet(X, A) \rightarrow C_\bullet(X, V)$$

which induce short exact sequence of abelian groups

$$0 \longrightarrow C_n(V, A) \longrightarrow C_n(X, A) \longrightarrow C_n(X, V) \longrightarrow 0$$

for each n .

- (b) For any triple $A \subset V \subset X$, show there exists a long exact sequence of relative homology groups as follows:

$$\cdots \longrightarrow H_n(V, A) \longrightarrow H_n(X, A) \longrightarrow H_n(X, V) \xrightarrow{\partial} H_{n-1}(V, A) \longrightarrow \cdots$$

(for instance, by citing a previous problem). This is called the *long exact sequence associated to a triple*, or the *long exact sequence of a triple*.

8. Deformation retracts

- (a) Recall that a *deformation retraction* of V to A is a map

$$F : V \times [0, 1] \rightarrow V$$

where

- (i) $F(a, t) = a$ for all $a \in A, t \in [0, 1]$,
- (ii) $F(v, 0) = v$ for all $v \in V$, and
- (iii) $F(v, 1) \in A$ for all $v \in V$.

Show that if V admits a deformation retraction to A , V is homotopy equivalent to A .

If such a deformation retraction exists, we say that A is a *deformation retract* of V .

- (b) Prove that if $A \subset V \subset X$ and A is a deformation retract of V , then there is an isomorphism of homology groups $H_n(X, A) \cong H_n(X, V)$ induced by the obvious map of pairs $f : (X, A) \rightarrow (X, V)$ given by $f(x) = x$. (“Triples” should be the obvious hint here.)

The upshot is that you can compute relative homology of (X, A) by replacing it with (X, V) , and vice versa.

- (c) Show that a deformation retraction from V to A induces a deformation retraction from the space V/A to the space A/A (i.e., a point).

9. Proof of the main theorem

Assume that $A \subset V \subset X$ is a sequence of subspaces where A is closed, and V is an open set which deformation retracts onto A .

- (a) Show there is a commutative diagram of a pairs of spaces

$$\begin{array}{ccc} (X, A) & \xrightarrow{f} & (X, V) \\ \downarrow & & \downarrow q \\ (X/A, A/A) & \xrightarrow{g} & (X/A, V/A) \end{array}$$

where the horizontal maps are given by $f(x) = x$ and $g([x]) = [x]$.

(b) By functoriality of relative homology, we now have a commutative diagram of abelian groups

$$\begin{array}{ccc} H_n(X, A) & \longrightarrow & H_n(X, V) \\ \downarrow & & \downarrow q_* \\ H_n(X/A, A/A) & \longrightarrow & H_n(X/A, V/A) \end{array}$$

for all n .

(c) By using the previous problems, conclude that the leftmost vertical arrow is an isomorphism.

10. Naturality of long exact sequences

This problem is also a problem purely in algebra.

(a) Suppose we have a commutative diagram of chain complexes as follows:

$$\begin{array}{ccccc} A & \xrightarrow{f} & B & \xrightarrow{g} & C \\ \downarrow \alpha & & \downarrow \beta & & \downarrow \gamma \\ A' & \xrightarrow{f'} & B' & \xrightarrow{g'} & C' \end{array}$$

Prove (for instance, by citing homework) that for every n , this induces a commutative diagram on homology groups

$$\begin{array}{ccccc} H_n A & \xrightarrow{f_*} & H_n B & \xrightarrow{g_*} & H_n C \\ \downarrow \alpha_* & & \downarrow \beta_* & & \downarrow \gamma_* \\ H_n A' & \xrightarrow{f'_*} & H_n B' & \xrightarrow{g'_*} & H_n C' \end{array}$$

(b) Assume that the maps f, g, f', g' induce short exact sequences

$$0 \longrightarrow A_n \xrightarrow{f_n} B_n \xrightarrow{g_n} C_n \longrightarrow 0 \quad \text{and} \quad 0 \longrightarrow A'_n \xrightarrow{f'_n} B'_n \xrightarrow{g'_n} C'_n \longrightarrow 0$$

for every n . Prove that there is a commutative diagram

$$\begin{array}{ccc} H_n C & \xrightarrow{\partial_n} & H_{n-1} A \\ \downarrow \gamma_* & & \downarrow \alpha_* \\ H_n C' & \xrightarrow{\partial'_n} & H_{n-1} A' \end{array}$$

where ∂_n and ∂'_n are the maps constructed in Problem 1.

- (c) Conclude that the long exact sequences associated to f, g and to f', g' are *natural*. This just means that we have a commutative diagram as follows:

$$\begin{array}{ccccccccc} \cdots & \xrightarrow{\partial_{n-1}} & H_n A & \xrightarrow{f_*} & H_n B & \xrightarrow{g_*} & H_n C & \xrightarrow{\partial_n} & H_{n-1} A & \xrightarrow{f_*} & \cdots \\ & & \downarrow \alpha_* & & \downarrow \beta_* & & \downarrow \gamma_* & & \downarrow \alpha_* & & \\ \cdots & \xrightarrow{\partial'_{n-1}} & H_n A' & \xrightarrow{f'_*} & H_n B' & \xrightarrow{g'_*} & H_n C' & \xrightarrow{\partial'_n} & H_{n-1} A' & \xrightarrow{f'_*} & \cdots \end{array}$$

where each horizontal row is the short exact sequence from Problem 1.

- (d) Suppose we have a map of pairs $f : (X, A) \rightarrow (X', A')$. Show we have a commutative diagram of homology groups

$$\begin{array}{ccccccccc} \cdots & \xrightarrow{\partial_{n-1}} & H_n A & \xrightarrow{i_*} & H_n X & \longrightarrow & H_n(X, A) & \xrightarrow{\partial_n} & H_{n-1} A & \longrightarrow & \cdots \\ & & \downarrow f_* & & \downarrow f_* & & \downarrow f_* & & \downarrow f_* & & \\ \cdots & \xrightarrow{\partial'_{n-1}} & H_n A' & \xrightarrow{i'_*} & H_n X' & \longrightarrow & H_n(X', A') & \xrightarrow{\partial'_n} & H_{n-1} A' & \longrightarrow & \cdots \end{array}$$

Here, i_* is the map on homology induced by the inclusion $i : A \rightarrow X$, f_* is an abuse of notation (because it is used to denote many different maps), but every f_* is the map on homology induced by the map of pairs.