Homework 4: Mayer-Vietoris Sequence and CW complexes

Due date: Friday, October 4th.

0. Goals and Prerequisites

The goal of this homework assignment is to begin using the Mayer-Vietoris sequence and cellular homology to compute homology of basic shapes. Some proofs depend on algebraic manipulations, while others depend on geometric arguments. So this is a good first run at trying to do problems that are genuinely problems of algebraic topology. You may find many of these problems difficult, as some will require some geometric creativity. Don't be discouraged, in particular, if the problems about surfaces seem difficult. It takes some time and wisdom to develop the intuition for CW structures.

You need to know what CW homology is, and how to use the Mayer-Vietoris sequence. This can be found in the class notes.

1. Reduced homology

Recall that *reduced homology* $\tilde{H}_n(X)$ for a non-empty space X is defined to be

$$\tilde{H}_n(X) \cong H_n(X), \qquad \tilde{H}_0(X) \cong \ker(\epsilon / \operatorname{image}(\partial_1))$$

where ϵ is the map of abelian groups

$$C_0(X) \to \mathbb{Z}, \qquad \sum a_i x_i \mapsto \sum a_i.$$

If X is non-empty and $x_0 \in X$ is a basepoint, show

$$\tilde{H}_0(X) \cong H_0(X, x_0)$$

where the latter is relative homology.

2. Homology of wedges

Let X and Y be spaces, and choose a point $x_0 \in X$, $y_0 \in Y$ in each space. The *wedge sum* of the two spaces is defined to be the space

$$X \vee Y := X \bigcup_{x_0 \sim y_0} Y \cong (X \coprod Y)/x_0 \sim y_0.$$

This is the space obtained by gluing x_0 to y_0. For instance, if X and Y are both homeomorphic to S^1, their wedge sum is a figure 8. Show that

$$\widetilde{H}_n(X \vee Y) \cong \widetilde{H}_n(X) \oplus \widetilde{H}_n(Y)$$

for all n.

This is a sign that the *wedge sum* of pointed spaces is like *disjoint union* of unpointed spaces.

3. Spheres

In this problem you will prove the hairy ball theorem. Recall that the *degree* of a map $f : S^n \to S^n$ is defined as follows: Since f_* is a group map on $H_n(S^n) \cong \mathbb{Z}$, there is a unique integer d such that the map

$$f_* : H_n(S^n) \to H_n(S^n) \qquad \text{is given by} \qquad f_*(a) = d \cdot a$$

for any $a \in H_n(S^n)$. This d is called the *degree* of f.

(a) Using the Mayer-Vietoris sequence, and induction, compute all the homology groups of the n-sphere. Your argument needs to be more complete than the sketch I gave in class.

(b) Do this also using cellular homology.

(c) Fix $i \in \{0, \ldots, n\}$. Let $f_i : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ be the continuous map given by sending

$$(x_0, x_1, \ldots, x_i, \ldots x_n) \mapsto (x_0, x_1, \ldots, -x_i, \ldots x_n).$$

This is otherwise known as *reflecting* a point about the hyperplane $x_i = 0$. Show that $f_i|_{S^n}$ is a map of degree -1. You can do this using, for instance, the functoriality of cellular homology, or simplicial homology. You just need to find a convenient CW structure or simplicial structure.

(d) Define a map $g : S^n \to S^n$ by $g : \vec{x} \mapsto -\vec{x}$. Show g is a degree 1 map if n is odd, and otherwise a degree -1 map. This g is called the *antipodal* map.

(e) Show that the antipodal map is not homotopic to the identity of S^n if n is even.

(f) Using multiplication by i on $\mathbb{C}^m$, show that an odd-dimensional sphere $S^{2m-1} \subset \mathbb{C}^m$ always admits a nowhere-vanishing tangent vector field.

(g) Now prove this isn't possible for an even-dimensional sphere. That is, prove (using degree, for instance) that no even-dimensional sphere admits a tangent vector field that is nowhere vanishing. (It has to vanish somewhere!) It may be hard to think of a solution, so feel free to ask for a hint when you want it.

For $n = 2$, this is called the *hairy ball theorem*. For instance, it shows in physics that if you have any Gaussian surface (which is a fancy term for "surface") that is homeomorphic to a sphere, the electric field cannot be everywhere tangent to the surface *and* non-zero. So if you have, for instance, a metal sphere exposed to an electric field, there must be some point on the sphere where an electron there will not experience any force (assuming the electron is constrained to lie on the surface of the sphere).

Another common analogy is that "you can't comb a coconut" (because a totally flat comb-job would yield a tangent, nowhere-zero vector field). Another common slogan is that at any given point, somewhere on earth, the wind must not be blowing (if you disregard the components of wind that blow directly toward the sky).

4. Torsion homology

In this example we'll see our first example of *torsion* homology groups. We will construct a CW complex X with a single cell in dimensions 0, 1, and 2. (And no other cells). Note that X^0 and X^1 are uniquely determined, but there is ambiguity in X^2 depending on the attaching map $\Phi : \partial D^2 \to X^1$.

Recall that the *degree* of a continuous map $f : S^n \to S^n$ is defined as follows: f induces a map $H_n(S^n) \to H_n(S^n)$ on top homology—i.e., a map $\mathbb{Z} \to \mathbb{Z}$. But any group map $\mathbb{Z} \to \mathbb{Z}$ is determined completely by the image of $1 \in \mathbb{Z}$, and the image, $d \in \mathbb{Z}$, is called the *degree* of the continuous map f. (This is the same definition as above.)

(a) Fix an integer $d \in \mathbb{Z}$. Show that the map $f_d : e^{i\theta} \mapsto e^{id\theta}$ is a degree d map from S^1 to itself.

(b) Let the map Φ be given by f_d, and X the resulting CW complex. Compute the homology groups of X. (This depends on d.)

(c) (*) Show that X is not a manifold for $d \neq 2$. (For $d = 1$, it is not a manifold, but a manifold *with boundary*.)

5. Cylinders, Cones and Suspensions

Fix a non-empty, topological space X.

(a) Compute the homology of the cylinder (i.e., the product space) $X \times [0, 1]$, given the homology of X.

(b) Let CX be the space $X \times [0, 1]/\sim$, where $\sim$ is the equivalence relation $(x, 0) \sim (x', 0)$ for all $x, x' \in X$. (This means the subspace $X \times \{0\}$ is collapsed to a single point.) This space is called the *cone* on X. Show that CX is contractible.

Let SX be the space $CX/\sim$, where $\sim$ is the equivalence relation $(x, 1) \sim (x', 1)$ for all $x, x' \in X$. (This means the subspace $X \times \{1\}$ is collapsed to a single point.) This space is called the *suspension* of X. Put another way it is obtained from $X \times [0, 1]$ by collapsing each copy of X at the endpoints of $[0, 1]$ to a point (one point for each copy).

(c) If X is the n-sphere, show that SX is homeomorphic to the $(n + 1)$-sphere.

(d) Now X can be any space. Compute the homology of SX in terms of the homology of X.

(e) For each $n \in \mathbb{Z}_{\geq 0}$, fix an abelian group K_n. Show that there exists a space X such that

$$\tilde{H}_n(X) \cong K_n$$

for all n.

6. Homology of the n-torus

The n-torus is the space $T^n = S^1 \times \ldots \times S^1$; i.e., the n-fold product of a circle. So T^1 is just a circle, and T^2 is the surface of a doughnut. We'll see how to compute the homology of product spaces easily using the Kunneth formula later on, but for now we'll compute it using Mayer-Vietoris.

(a) Find an open cover of S^1 by two intervals, I_1 and I_2, such that the intersection $I_1 \cap I_2$ is equal to the disjoint union $J_1 \coprod J_2$ of two smaller intervals.

(b) Use the Mayer-Vietoris sequence for the open cover $U = I_1 \times T^{n-1}$, $V = I_2 \times T^{n-1}$, and $U \cap V = (J_1 \sqcup J_2) \times T^{n-1}$. By induction on n, conclude that

$$H_k(T^n) = \mathbb{Z}^{\binom{n}{k}}$$

where $\binom{n}{k}$ is the binomial coefficient "n choose k". In other words, if you look at the rank of the kth homology group of the n-torus, you will see the kth coefficients on the nth row of Pascal's triangle.

7. Homology of surfaces using CW structure

Fix a number $g \geq 0$. and let X^1 be the wedge sum of $2g$ circles all connected at some basepoint $x_0 \in X^1$. We label the circles by $a_1, b_1, \ldots, a_g, b_g$. (So for instance, if $g = 2$, there are four circles, with names a_1, b_1, a_2, b_2.)

Note that X^1 is a CW complex with a single 0-cell, and $2g$ 1-cells. By abuse of notation, we also let

$$a_i : [0,1] \to X^1, \qquad b_i : [0,1] \to X^1, \qquad a_i(0) = a_i(1) = b_i(0) = b_i(1) = x_0$$

denote the composite maps

$$D^1 \to (\coprod_\alpha D^1_\alpha \coprod X^0) \to X^1$$

defining the 1-cell labeled by a_i and b_i. Here, the first map is just including the 1-disk to the cell D^1_α corresponding to the loop a_i, and the second map is the quotient map that defines the 1-skeleton of X^1.

Informally, a_i "is" the map defining the loop a_i, and likewise for b_i. Note that these generate the homology group $H_1(X^1) \cong \mathbb{Z}^{2g}$. Finally, by the symbol a_i^{-1}, we mean the path going the other direction: It is the map

$$a_i^{-1} : [0,1] \to X^1, \qquad a_i^{-1}(t) := a_i(1-t), \qquad t \in [0,1].$$

And likewise,

$$b_i^{-1}(t) := b_i(1-t).$$

For instance, one can show that the 1-simplex a_i^{-1} is the additive inverse of a_i in H_1, and likewise for b_i.

Fix a disk D^2, and think of its boundary $\partial D^2 \cong S^1$ as an interval $I = [0, 4g]$ (so it has length $4g$), modulo the endpoints. Then there's a map

$$\Phi : S^1 \to X^1$$

given as follows: For any integer $n = 0, \ldots, 4g-1$, write $n = 4j + l$ where j is an integer and l is an integer between 0 and 3, inclusive. Then we define

$$\Phi(n+t) = \begin{cases} a_j(t) & \text{if } l = 0 \\ b_j(t) & \text{if } l = 1 \\ a_j^{-1}(t) & \text{if } l = 2 \\ b_j^{-1}(t) & \text{if } l = 3 \end{cases}, \qquad \text{where } t \in [0,1] \text{ and } n+t \in [0,4g].$$

In other terms, we say that the map $S^1 \to X^1$ is given by the path $b_g^{-1}a^{-1}b_g a_g \ldots b_1^{-1}a_1^{-1}b_1 a_1$. One way to think of this is to think of ∂S^1 as a $4g$-gon, and to label the sides of the $4g$-gon by the symbols a_i, b_i and their inverses.

See for instance, the images on Page 5 of Hatcher, on the page before Example 0.1.

Let $X = X^2$ be the CW given by gluing D^2 onto X^1 by Φ.

(a) Draw a picture for $g = 1$ and $g = 2$ showing how X is the surface of genus g.

(b) Compute the homology groups of X using cellular homology.

8. Homology of the punctured torus

Think of the torus T^2 as the quotient space obtained by taking $I^2 = [0,1] \times [0,1]$ and modding out by the relation

$$(t,0) \sim (t,1) \qquad \text{and} \qquad (0,t) \sim (1,t) \qquad \text{for any } t \in [0,1].$$

Note this is actually the same description that was given for the torus above, where we called it the surface of genus 1.

Fix a point x of T^2—for instance, the point $(1/2, 1/2)$ in the interior of I^2.

(a) Consider the (no longer compact) space $T^2 - \{x\}$ obtained by puncturing the torus; i.e., by removing the point. Show it is homotopy equivalent to the wedge sum of two circles. A proof by picture will be accepted only if the picture is very-well drawn and convincing. State the homology of the punctured torus; explain your answer.

(b) Consider the shape X obtained by deleting a small open disk around x; this is called the *surface of genus 1 with one boundary circle*. (The boundary circle is the boundary of the open disk you deleted.) Show that this (compact) space is homotopy equivalent to the (non-compact) space from Part (a). Again, a picture can suffice so long as it is carefully drawn.

(c) Show that the inclusion $i : S^1 \to X$ of the boundary circle is an isomorphism on H_0, but is the zero map on H_1. No picture will be accepted as a full answer, but you can draw one for intuition.

(d) Draw a picture X embedded in $\mathbb{R}^3$. Draw a set of curves that generate $H_1(X)$.

9. Homology of the n-punctured torus

(a) Let X be the space obtained by putting two punctures into T^2. Show that $H_1(X)$ is isomorphic to $\mathbb{Z}^3$.

(b) Draw three curves that generate H_1.

(c) If you haven't already, prove that the twice-punctured torus is homotopy equivalent to a *bouquet of 3 circles*. This is a fancy term for a wedge sum of 3 circles.

(d) More generally, prove that the n-punctured torus is homotopy equivalent to a bouquet of $(n+1)$ circles.

(e) Show that a wedge of n circles is not a manifold. Explain (in one sentence) why this shows that "being a manifold" is not preserved under homotopy equivalence.

(f) Draw an n-punctured torus and draw the $n+1$ generators of H_1.

10. Homology of surfaces using Mayer-Vietoris

The sphere is a surface of genus 0. The torus is a surface of genus 1. Attaching more handles, one gets a surface of genus g. (For instance, the surface of a pretzel is a surface of genus 3.) If we remove n small open disks from a surface of genus g, we obtain a surface of the same genus, but with n boundary circles. Let $\Sigma_{g,n}$ denote the surface of genus g with n boundary circles. (Note this is homotopy equivalent to simply removing n points from Σ_g.)

(a) Compute the homology groups of $\Sigma_{g,0}$ for $g \geq 0$ by Mayer-Vietoris and the previous exercise, or otherwise.

(b) Compute the homology groups of $\Sigma_{g,n}$.