Homework 5: Cellular Approximation

Due date: Friday, October 11th.

0. Background and Goals

0.1. Notation and definitions. Let X be a CW complex. In class we have denoted by A_n the set of n-cells in X. **From now on, we will use the notation $\mathcal{A}_n$ to denote the set of n-cells.** This is to distinguish it from a subspace $A \subset X$. If there are a few CW complexes floating around, we'll write $\mathcal{A}_n^X$ to mean the set of n-cells of a CW complex X, and $\mathcal{A}_n^A$ to mean the set of n-cells of a CW complex A, for instance.

DEFINITION 1. *A subspace $A \subset X$ is a* sub CW complex *if A is closed and is a union of cells in X. A choice of (X, A) is called a* CW pair.

In other words, any sub CW complex A determines some subset $\mathcal{A}_n^A \subset \mathcal{A}_n^X$ for each n, and can be written as the space

$$A = \bigcup_{\alpha \in \mathcal{A}_n^A, n \geq 0} q(\Phi_\alpha(D_\alpha^n)).$$

Here, q is the quotient map

$$q : (\coprod_\alpha D_\alpha^n) \coprod X^{n-1} \to X^n.$$

Note in particular A is a CW complex; we denote its n-skeleton by A^n.

EXAMPLE 0.1.
1. If $A = \emptyset$ then (X, A) is a CW pair for any CW complex X.
2. If $A = * \in X^0$ is a point in the 0-skeleton of X, then (X, A) is a CW pair.
3. Recall that $\mathbb{C}P^n$ is built out of a single cell in each even dimension. The pair $(\mathbb{C}P^n, \mathbb{C}P^k)$ is a CW pair for $k \leq n$.
4. We can give any polyhedron a CW structure—a 0-cell for every vertex, a 1-cell for every edge, and a 2-cell for every face. (Regardless of whether the face is a square, a triangle, or whatever.) Then if A is the closure of a face (or a union of such closures) or a choice of edges (including the boundaries of the edges) or just a collection of vertices of the polyhedron, or a union of any of these options, then (X, A) is a CW pair.

Note that not every choice of $\mathcal{A}_n^A \subset \mathcal{A}_n^X$ makes sense as a subspace (or a space at all!), since the higher cells we choose must glue onto A^{n-1}.

DEFINITION 2. *Let $\alpha \in \mathcal{A}_n$. Then by e_α^n, we mean the open n-cell of X corresponding to α.*

Concretely, we have the map

$$D_\alpha^n \to (\coprod_{\alpha \in \mathcal{A}_n} D_\alpha^n) \coprod X^{n-1} \to X^n$$

where the first map is including D^n_α into the disjoint union, and the second map is collapsing via the gluing maps Φ^n_α. Then e^n_α is the image of interior(D^n_α) under this composition. It is homeomorphic to the open n-disk.

Definition 3. *Let (X, A) and (Y, B) be CW pairs. A map of pairs $f : (X, A) \to (Y, B)$ is called* cellular *if*

$$f(A^n) \subset B^n \qquad \text{and} \qquad f(X^n) \subset Y^n$$

for each n. In particular, if $A = B = \emptyset$, a continuous map $f : X \to Y$ is called cellular if $f(X^n) \subset f(Y^n)$ for all n.

In this homework you will prove a fundamental result for modern topology: Cellular approximation.

Theorem 0.2 (Cellular approximation theorem). Let (X, A) and (Y, B) be CW pairs. If $f : (X, A) \to (Y, B)$ is a continuous map of pairs, then f is homotopic to a cellular map of pairs. That is, the homotopy can be chosen so that f_t is a map of pairs for each t, and f_1 is a cellular map.

For instance, if Y is a space that can be given a CW structure with no cells in dimensions $1 \leq k \leq N$, then any map from a k-dimensional connected X, with $1 \leq k \leq N$, must be null-homotopic (i.e., homotopic to a constant map). This follows by taking $A = B = \emptyset$ and observing that a cellular map must take $X = X^k$ to some vertex of Y.

1. Homotopy Extension Property

The strategy for constructing the homotopy f_t will be to build it step by step on each n-skeleton. That is, we'll first construct cellular maps on X^n. Then we'll have to extend a homotopy defined only on X^n to a homotopy defined on all of X! This turns out to be not too difficult, and we prove it here.

Definition 4 (Homotopy Extension Property). *We say that a pair of spaces (X, A) satisfies the* homotopy extension property*, or* satisfies HEP*, if for any map $f : (X, A) \to Y$, and any homotopy*

$$F^A : A \times [0, 1] \to Y \qquad \text{such that} \qquad F^A(a, 0) = f(a)$$

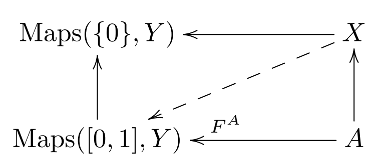
there exists a homotopy

$$F : X \times [0, 1] \to Y$$

such that

$$F(x, 0) = f(x) \qquad \text{and} \qquad F(a, t) = F^A(a, t) \qquad \forall a \in A.$$

Diagrammatically, this means the dashed arrow exists:



Here, we write F^A because a map $A \times [0, 1] \to Y$ is the same thing as a map from A to Maps$([0, 1], Y)$.

As intuition, this means that if you fix a map from $X \to Y$, and then you wiggle what the map does on A, then you can find a wiggle on all of X extending the wiggle on A.

(a) Assume (X, A) has the homotopy extension property. Show that the space

$$X \times [0,1]$$

retracts onto the subspace

$$A \times [0,1] \bigcup X \times \{0\}.$$

(Hint: Take Y to be this subspace, and f to be the map embedding $f(x) = (x, 0)$. Also, take F^A to be the embedding of $A \times [0,1]$ into Y.)

(b) Conversely, suppose that $X \times [0,1]$ retracts onto the subspace from above. Show that (X, A) has HEP.

2. Extending wiggles of X^{n-1} to wiggles of X^n

(a) Consider the space $D^n \times [0,1] \subset \mathbb{R}^{n+1}$, and the subspace

$$\partial D^n \times [0,1] \bigcup D^n \times \{0\}.$$

Show that $D^n \times [0,1]$ deformation retracts onto this subspace.

(Hint: Consider the disk as sitting on the plane of height 0 inside $\mathbb{R}^n \times \mathbb{R}$. Consider point $p = (0,2) \in D^n \times \mathbb{R} \subset \mathbb{R}^n \times \mathbb{R}$, which sits above the origin of the disk at height 2. If it looks down, it can see (the top of) $D^n \times [0,1] \subset \mathbb{R}^n \times \mathbb{R}$ sitting below it. What do you get if you start drawing rays from p to points in $D^n \times [0,1]$?)

(b) Let F^n_t be your deformation retraction from (a). For each $n = 0, 1, 2$, draw the image of F^n_t for $t = 0, t = 1/2$, and $t = 1$. (Note that if $n = 0$, then $\partial D^n = \emptyset$.)

(c) Does the pair $(D^n, \partial D^n)$ have the homotopy extension property? (Hint: Problem 1.)

(d) In (a), you constructed a map $F^n : [0,1]_t \times (D^n \times [0,1]) \to D^n \times [0,1]$ exhibiting a deformation retraction. The subscript t is to simply remind you that $[0,1]_t$ parametrizes the homotopy. (Unlike our usual notation, I'm putting the time parameter to the left; this is to make sure we don't confuse it with the parameter for the unit interval in $D^n \times [0,1]$.)

Let $\alpha \in \mathcal{A}^X_n$. Show that

$$F^{n,\alpha}(t, x, s) := \begin{cases} (\Phi_\alpha \times \mathrm{id}_{[0,1]}) \circ F^n(t, \Phi_\alpha^{-1} x, s) & x \in e^n_\alpha \\ (x, s) & \text{otherwise} \end{cases} \tag{1}$$

exhibits a deformation retract of $X^n \times [0,1]$ onto the space

$$(X^n - e^n_\alpha) \times [0,1] \bigcup X^n \times \{0\}.$$

(e) Draw this retraction for $n = 2$ if X is the unit 2-disk with a single cell in dimensions 0,1, and 2. (For instance,X^1 is just a circle, and $X^2 = X$ is obtained by attaching the disk in the obvious way.) Draw the deformation retraction from part (d) for times $t = 0, 1/2, 1$.

(f) Does the pair $(X^n, X^n - e^n_\alpha)$ have the homotopy extension property? (Hint: Problem 1.)

(g) Show that the pair (X^n, X^{n-1}) has HEP.
(Hint: Note that the space

$$X^n - (\bigcup_{\alpha \in \mathcal{A}^X_n} e^n_\alpha)$$

is by definition equal to the space $X^{n-1} \subset X^n$. So can you write a piecewise formula for "Perform $F^{n,\alpha}$ simultaneously for all $\alpha \in \mathcal{A}^X_n$"? It only needs a minor modification of (1).)
You've proven that if we wiggle a homotopy on X^{n-1}, we can wiggle it on all of X^n, too.

3. Extending wiggles of A to wiggles of X.

Now, given a CW pair (X, A), our goal is to show that (X, A) has HEP. By Problem One, this means we'll exhibit a retraction from $X \times [0, 1]$ to the space

$$X \times \{0\} \bigcup A \times [0, 1].$$

We do this one dimension at a time. So given $n \geq 0$, define a function

$$[0, 1]_t \times X^n \times [0, 1] \to X^n \times [0, 1]$$

as follows:

$$(t, x, s) \mapsto \begin{cases} (x, s) & x \in X^k, \quad k \leq n - 1 \\ (x, s) & x \in e^n_\alpha, \quad \alpha \in \mathcal{A}^A_n \\ F^{n,\alpha}(t, x, s) & x \in e^n_\alpha, \quad \alpha \notin \mathcal{A}^A_n, \quad \alpha \in \mathcal{A}^X_n. \end{cases}$$

The interpretation is: "If x is in a lower dimension, don't move it. If it's a point in A, don't move it. If it's a point in dimension n, and it ain't in A, wiggle the point (x, s) so it falls into $A \times [0, 1]$ or into $X^n \times \{0\}$."

(a) For $n \geq 1$, show that this exhibits a deformation retraction from $X^n \times [0, 1]$ to

$$X^{n-1} \times [0, 1] \bigcup A^n \times [0, 1] \bigcup X^n \times \{0\}.$$

(b) For $n \geq 1$, does the pair $(X^n, A^n \cup X^{n-1})$ satisfy HEP?

(c) What happens when you apply the same formula as above when $n = 0$?

(d) Does the pair (X^0, A^0) satisfy HEP?

Believe it or not, the following definitions are for notational convenience. Choose some reparametrization of the time interval $[0,1]_t$ by an interval $[\frac{1}{2^{n+1}}, \frac{1}{2^n}]$. That is, a homeomorphism

$$\phi_n : [\frac{1}{2^{n+1}}, \frac{1}{2^n}] \cong [0,1] \qquad \phi_n(\frac{1}{2^{n+1}}) = 0, \qquad \phi_n(\frac{1}{2^n}) = 1.$$

Then for each n, we define a map

$$r^n : [\frac{1}{2^{n+1}}, \frac{1}{2^n}]_t \times X^n \times [0,1] \to X^n \times [0,1]$$

as follows:

$$(t,x,s) \mapsto \begin{cases} (x,s) & x \in X^k, \qquad k \leq n-1 \\ (x,s) & x \in e^n_\alpha, \qquad \alpha \in \mathcal{A}^A_n \\ F^{n,\alpha}(\phi_n(t),x,s) & x \in e^n_\alpha, \qquad \alpha \notin \mathcal{A}^A_n, \qquad \alpha \in \mathcal{A}^X_n. \end{cases}$$

The interpretation of r^n is: "Do the deformation retraction from part (a), but really quickly, during the time interval $[\frac{1}{2^{n+1}}, \frac{1}{2^n}]$." The reason we're doing this is because X and A may be infinite-dimensional, so it would take forever to retract $X \times [0,1]$ if we did our previous retractions one unit interval at a time. This reparametrization will allow us to perform the infinite number of retractions r^n in a finite amount of time.

Now define a map

$$r : [0,1]_t \times X \times [0,1] \to X \times [0,1]$$

by the formula

$$(t,x,s) \mapsto \begin{cases} (x,s) & x \in X^n, t \in [0, \frac{1}{2^{n+1}}] \\ r_n(t,x,s) & x \in X^n, t \in [\frac{1}{2^{n+1}}, \frac{1}{2^n}] \\ r_{n-1}(t, r_n(\frac{1}{2^n}, x, s)) & x \in X^n, r_n(\frac{1}{2^n}, x, s) \in X - A, t \in [\frac{1}{2^n}, \frac{1}{2^{n-1}}] \\ r_k(t, r_{k+1}(\frac{1}{2^{k+1}}, \ldots, r_{n-1}(\frac{1}{2^{n-1}}, r_n(\frac{1}{2^n}, x, s))) \ldots) & x \in X^n, r_{k+1}(\frac{1}{2^{k+1}}, x, s) \in X - A, \\ & \quad t \in [\frac{1}{2^n}, \frac{1}{2^k}], \qquad k < n. \\ (a,s) & a \in A, t \in [0,1] \end{cases}$$

The first line means: If you're in the n-skeleton, don't move until your time comes. You let the kids in the higher-dimensional cells move first.

The second line tells you when your time has come: If t is bigger than $\frac{1}{2^{n+1}}$, your time has come. So get moving, and retract just as r_n tells you to do.

The third line is actually redundant in light of the fourth line, but I wrote it in to make things easier to parse. It says that if you're in the n-skeleton, once your time comes, start retracting according to r_n. Well, as t moves forward in time, and if you're still not in A, you can't rest! You must keep retracting. This third line says, move according to the next retraction, called r_{n-1}.

The fourth line is hard to parse, but it's really just an iteration of the third line as t keeps moving forward. The fourth line means that if x begins in the n-skeleton, but t is a time all the way forward in time, then retract x into $A \times [0,1] \bigcup X^{n-1}$ using the retraction r_n, then if you're still not in A,retract it further using the retraction r_{n-1}, and so forth until you are finally retracting via r_k.

The last line means that if you're in $A \times [0,1]$, don't move. (The third and fourth lines don't tell you what to do at a big time t if you're in a higher cell of A. This is to cover for that.)

The interpretation is: "Okay, do all the deformation retractions in succession, really quickly."

Note that every point $t \in [0,1]$ is in the interval $[0, \frac{1}{2^k}]$ for some k small enough, so $r(t,x,s)$ is defined for every point (t,x,s) in the domain.

We'll verify that it's continuous in the next problem. For now:

(e) Show that $r(0,x,s) = (x,s)$. This means r_0 is the identity map for $X \times [0,1]$.

(f) Show that $r(1,x,s)$ is contained in the space

$$A \times [0,1] \bigcup X \times \{0\}.$$

(g) Show that $r(t,a,s) = (a,s)$ for all $(a,s) \in A \times [0,1]$. Show that the above map is well-defined; that is, that the last line does not conflict with the rest of the lines.

Pending the continuity of r, you've shown that $X \times [0,1]$ deformation retracts onto $A \times [0,1] \bigcup X \times \{0\}$. By Problem 1, you've shown that any CW pair (X,A) satisfies HEP.

4. Continuity of r

Recall that a CW complex

$$X = \bigcup_{n \geq 0} X^n$$

is topologized so that $U \subset X$ is open if and only if $U \cap X^n$ is open in X^n for all $n \geq 0$.

(a) You may assume Q is a line interval. Show that a function

$$f : X \times Q \to Y$$

is continuous if and only if the restriction

$$f|_{X^n \times Q} : X^n \times Q \to Y$$

is continuous for all n.

(b) Show that the function r from the previous problem is a continuous map.

5. Deforming maps into Y, I

(a) For $n \geq 1$, let $q \in \text{interior}(D^n)$ be an interior point of the standard n-disk. Show that there exists a deformation retract from $D^n - \{q\}$ to ∂D^n.

(b) Let $\mathcal{A}_n^Y$ be the set of n-cells of a CW complex Y. For all $\beta \in \mathcal{A}_n$, choose some $q_\beta \in e_\beta^n \subset Y^n$. Show there exists a deformation retraction from $Y^n - \{q_\beta\}$ to Y^{n-1}.

In other words, if you poke a hole in every top-dimensional cell of Y^n, you can contract the resulting space to the $(n-1)$-skeleton.

(c) Let Z be a space, and $g : Z \to Y^n$ a continuous map such that

$$\text{for all } \beta \in \mathcal{A}_n^Y, \text{ there is some } q_\beta \in e_\beta^n \text{ for which } q_\beta \text{ is not in the image of } g.$$

Show that g is homotopic to a map whose image is contained in Y^{n-1}, and that this homotopy g_t can be chosen in such a way that if $g(x) \in Y^{n-1}$, then $g_t(x) = g(x)$ for all times t.

6. Deforming maps into Y, II

In this problem, you may assume the following Lemma:

LEMMA 6.1. Let $n < k$. Then every continuous map from D^n to D^k is homotopic to a map relative to the boundary of D^n which is not a surjection. More generally, if Z is obtained by attaching a k-cell to some space Z', then any map $f : D^n \to Z$ can be homotoped to a map which does not surject onto the k-cell.

This is obvious when $n = 0$. For $n \neq 0$, it turns out one can find maps $D^n \to D^k$ which *are* surjective. (Google for instance "space-filling curve".) We'll assume the Lemma for now.

Fix a map $f : X \to Y$, where X and Y are CW complexes. Also fix a sub complex $A \subset X$, and assume that $f(A^n) \subset Y^n$ for all n.

Consider the map

$$f|_{X^0} : X^0 \to Y.$$

Since $Y = \bigcup_{k \geq 0}$, each $x \in X^0$ has image $f(x) \in Y^{k_x}$ for some $k_x \in \mathbb{Z}_{\geq 0}$. (Note that if X^0 is infinite, the k_x may grow arbitrarily large.) Recall by definition of CW complex that every connected component of Y must contain some element of Y^0.

(a) Using Problem 5 repeatedly, exhibit a path $\gamma_x : [0,1] \to Y$ from $f(x)$ to some $y \in Y^0$. If $f(x) \in Y^0$, obviously you can choose the constant path.

(b) Exhibit a homotopy from $f|_{X^0} : X^0 \to Y$ to a map $g^0 : X^0 \to Y$ such that $g(X^0) \subset Y^0$. Show that you can choose this homotopy to be constant on A^0, so $G(a,t) = g^0(a) = f(a)$ for all $a \in A^0$.

(c) Show that there exists a homotopy from $f : X \to Y$ to some function $f^0 : X \to Y$ such that $f^0(X^0) \subset Y^0$. (Hint: Note that for any n, (X, X^n) is a CW pair. What do Problems 3 and 4 tell you? And why did I have you make g^0?) Show that this can be chosen to be constant on A.

(d) Assume by induction that we have defined a map

$$f^{n-1} : X \to Y$$

such that (i) f^{n-1} is homotopic to f, (ii) $f^{n-1}(X^{n-1}) \subset Y^{n-1}$, and (iii) the homotopy from f to f^{n-1} is constant on A.

Recall that since D^n is compact, the composite map

$$D^n_\alpha \to X \to Y$$

must meet only finitely many cells of Y. Hence for all $\alpha \in \mathcal{A}^X_n$, $f^{n-1}(e^n_\alpha)$ is contained in Y^{k_α} for some $k_\alpha \in \mathbb{Z}_{\geq 0}$.

Show by repeated use of Problem 5 and the Lemma (on all $\alpha \in \mathcal{A}^X_n$ at once) that $f^{n-1}|_{X^n}$ is homotopic to a map $g^n : X^n \to Y$ such that $g^n(X^n) \subset Y^n$. Show that this can be chosen so the homotopy is constant on A^n.

(e) Show there exists a map $f^n : X \to Y$, homotopic to f^{n-1}, such that $f^n(X^n) \subset Y^n$. (Hint: HEP.) Show this can be chosen so that the homotopy from f^n to f^{n-1} is constant on A.

(f) Show that if X is a finite-dimensional CW complex (so $X = X^n$ for some finite n), then any continuous map $f : X \to Y$ can be made cellular. Further show that if f is cellular on a subcomplex $A \subset X$, then the homotopy can be chosen to be constant on A.

7. Deforming maps into Y, III

Now we take care of the case when X is not finite-dimensional, using the same trick as in Problem 3. Let

$$R_n : [1 - \frac{1}{2^n}, \frac{1}{2^{n+1}}] \times X \to Y$$

be the homotopy from f^{n-1} to f^n, but accelerated. (For instance, by using a variant of the parametrization ϕ_n from before.) This means that

$$R_n(1 - \frac{1}{2^n}, x) = f^{n-1}(x) \qquad \text{and} \qquad R_n(\frac{1}{2^{n+1}}, x) = f^n(x)$$

where by f^{-1} we mean the original function f. Define a map

$$R : [0, 1] \times X \to Y$$

combining all the homotopies R_n using techniques similar to Problem 3.

(1) Show that $R(0, x) = f(x)$. This means R exhibits a homotopy from f (to some other function).

(2) Show that if $x \in X^n$, then $R(1, x) \in Y^n$.

(3) Show that $R(t, a) = f(a)$ for all $a \in A$.

Pending the continuity of r, you've exhibited a homotopy R from f to a cellular map.

(4) Using techniques similar to Problem 4, show that R is a continuous function.

8. Cellular Approximation for Pairs

Show that any map of pairs $(X, A) \to (Y, B)$ is homotopic to a cellular map of pairs. (Hint: Just do the non-pair case step by step, and use HEP!)

9. CW Structures on Products

Let X and Y be CW complexes. In this problem we will show that $X \times Y$ has a CW structure.

(a) Show that for two disks of dimensions a, b, we have

$$\partial(D^a \times D^b) \cong (\partial D^a) \times D^b \bigcup_{\partial D^a \times \partial D^b} D^a \times (\partial D^b).$$

The righthand side means we glue the two spaces

$$(\partial D^a) \times D^b \qquad \text{and} \qquad D^a \times (\partial D^b)$$

along the common subset $\partial D^a \times \partial D^b$.

It may be easiest to choose a homeomorphism $D^n \cong I^n$.

(b) We will put a CW structure on $X \times Y$ where the set of n-cells is given by

$$\mathcal{A}_n^{X \times Y} := \coprod_{a+b=n} \mathcal{A}_a^X \times \mathcal{A}_b^Y$$

where $0 \leq a \leq n$ and $0 \leq b \leq n$.

Recall that q_α is the map

$$D^n_\alpha \to (\coprod_{\alpha \in \mathcal{A}_n^X} D^n_\alpha) \coprod X \to X.$$

Let $(\alpha, \beta) \in \mathcal{A}_a^X \times \mathcal{A}_b^Y$. Show by induction that the maps

$$\Phi_\alpha \times q_\beta, \qquad q_\alpha \times \Phi_\beta$$

define an attaching map for the disk

$$D^n_{\alpha,\beta}.$$

You should use the previous part of the problem to choose a convenient description of $\partial D^n_{\alpha,\beta}$ onto $(X \times Y)^{n-1}$.

(c) Using the CW structure for S^1 consisting of one 0-cell and one 1-cell, how many k-cells are in the product CW space $(S^1)^n = S^1 \times \ldots S^1$? (Use the CW structure from the previous part of this problem.) As a hint, think of Pascal's triangle.

10. CW Structures on Mapping Cones

I won't give you as many hints for this one.

Let $f : X \to Y$ be a continuous map. Recall that the mapping cone of f is defined to be the quotient space

$$M_f := (X \times [0,1] \coprod Y)/\sim$$

where the equivalence relation we quotient by is

$$(x, 1) \sim f(x).$$

Show that if X and Y are CW complexes, and f is a cellular map, then M_f is a CW complex. Construct this CW structure in such a way that the pair $(M_f, X \times \{0\})$ is a CW pair.