Homework 6: Excision for homotopy groups

Due date: Friday, October 18th.

0. Goals and Assumptions: The homotopy excision theorem

DEFINITION 5. *Let $A \subset X$. The pair (X, A) is said to be n-*connected *if*

$$\pi_i(X, A) = 0$$

for all $i \leq n$, and if every path-component of X contains a point of A.

*We say the pointed space X is n-*connected *if (X, x_0) is n-connected as a pair.*

REMARK 0.1. For instance, if (X, A) is n-connected, then the map $\pi_0(A) \to \pi_0(X)$ is a surjection.

Throughout this homework, you should assume the following theorem:

THEOREM 0.2 (Homotopy excision theorem). Let X be a CW complex. Let A and B be sub-CW complexes such that (1) $A \cap B = C$ is a sub-CW complex, and (2) $X = A \cup B$. Finally, assume that (A, C) is a-connected, and (B, C) is b-connected. Then the map of CW pairs

$$(A, C) \to (X, B)$$

is an isomorphism on π_i for $i < a + b$, and is a surjection for $i = a + b$.

REMARK 0.3. This is in contrast to homology, where $H_*(X)$ was easy to compute given $H_*(A)$, $H_*(B)$, and $H_*(C)$.

In this homework, using the homotopy excision theorem, you will define the stable homotopy groups of spheres, and compute homotopy groups for quotients.

1. Wedge sum v. Disjoint union

Recall that given two pointed spaces X and Y, their *wedge sum* is the topological space

$$X \vee Y := (X \coprod Y)/x_0 \sim y_0$$

given by gluing x_0 to y_0.

(a) Let A, B and C be (non-pointed) topological spaces. Show that there is a bijection of sets

$$\mathrm{Maps}(A \coprod B, C) \cong \mathrm{Maps}(A, C) \times \mathrm{Maps}(B, C)$$

where Maps as usual denotes the set of continuous maps. (By the way, one can actually prove this is a homeomorphism of *spaces* but we don't define a topology on Maps.)

(b) Let X, Y and Z be pointed spaces. Assume that the above bijection of sets is actually a homeomorphism of spaces.

Further assume the following property: Fix $A \subset X$ and let $\text{Maps}_A(X, Y)$ be the space of continuous maps $f : X \to Y$ that factor through X/A (so $f(A)$ is a point). Then the two spaces

$$\text{Maps}_A(X, Y) \cong \text{Maps}(X/A, Y)$$

are homemorphic.

Give the space of pointed maps the subspace topology, inherited from the space of all continuous maps. Show that there is a homeomorphism of spaces

$$\text{Maps}_*(X \vee Y, Z) \cong \text{Maps}_*(X, Z) \times \text{Maps}_*(Y, Z).$$

2. Smash product versus product

Given two pointed spaces X and Y, we define their *smash product* to be the space

$$X \wedge Y := (X \times Y)/X \vee Y.$$

Here's another way of writing this: Any point (x, y_0) is identified with the point (x_0, y_0). Likewise, any point (x_0, y) is identified with (x_0, y_0).

(a) Let A, B and C be sets. Show that there is a bijection of sets

$$\text{Maps}(A \times B, C) \cong \text{Maps}(A, \text{Maps}(B, C))$$

where Maps denotes the set of maps.

(b) Let X, Y and Z be pointed spaces. Assume the homeomorphism of spaces, analogous to the bijection above

$$\text{Maps}(A \times B, C) \cong \text{Maps}(A, \text{Maps}(B, C)).$$

Give the space of pointed maps the subspace topology, inherited from the space of all continuous maps. Show that there is a homeomorphism of spaces

$$\text{Maps}_*(X \wedge Y, Z) \cong \text{Maps}_*(X, \text{Maps}_*(Y, Z)).$$

(c) Recall that for any triplet of spaces, we have the homeomorphism

$$A \times (B \coprod C) \cong (A \times B) \coprod (A \times C).$$

If X, Y, Z are pointed spaces, prove that there is a homeomorphism of pointed spaces

$$X \wedge (Y \vee Z) \cong (X \wedge Y) \vee (X \wedge Z).$$

(d) (*Bonus) This can be a tedious problem. If A, B, C are all compactly generated and weakly Hausdorff, show that the bijection $\text{Maps}(A \times B, C) \cong \text{Maps}(A, \text{Maps}(B, C))$ above is a homeomorphism using the compact-open topology. I haven't defined these terms, so this is an "outside research project" for you to do. I would encourage tackling the rest of the homework problems before tackling this one.

3. The reduced suspension functor

Choose a basepoint on the unit circle S^1. Given a pointed space X, the space

$$\Sigma X := S^1 \wedge X$$

is called the *reduced suspension of* X. This is different from the suspension we defined two homeworks ago, but we use the same notation Σ for both the reduced and unreduced suspensions. Usually when X is pointed, one always takes the reduced suspension, and when X is not pointed, one simply takes the suspension.

In this homework we will always use the word "suspension" to really mean "reduced suspension". This is also a common convention in the world of pointed spaces.

(a) Fix an integer $n \geq 0$. Choose a basepoint on the n-sphere. Show there is a homeomorphism

$$\Sigma S^n \cong S^{n+1}.$$

That is, the suspension of a sphere is a sphere.

(b) Show that every map of pointed spaces $f : X' \to X$ induces a map of pointed spaces $\Sigma f : \Sigma X' \to \Sigma X$, induced by the map

$$S^1 \times X' \to S^1 \times X, \qquad (s, x') \mapsto (s, f(x')).$$

(c) Prove that Σ defines a functor from the category of pointed spaces to the category of pointed spaces.

(d) Prove that if $f : S^n \to S^n$ is a map of degree k, then $\Sigma f : S^{n+1} \to S^{n+1}$ is a map of degree k as well. (For instance, by choosing an appropriate CW structure on S^n and looking at the induced map on cellular homology.)

(e) Show that suspension induces a map of sets

$$\Sigma : \pi_j(X, x_0) \to \pi_{j+1}(\Sigma X, x_0)$$

for every pointed space X, and for every $j \geq 0$. Show that this is a group homomorphism for $j \geq 1$.

(f) Consider the snowcone

$$C'X := X \times I/(X \times \{0\} \cup \{x_0\} \times I).$$

It's otherwise known as $X \wedge I$ where the basepoint for I is chosen to be $0 \in I$. Note $C'X$ has a basepoint given by the equivalence class containing x_0, which we denote by $*$.

Fix a map

$$f : (I^j, \partial I^j) \to (X, x_0)$$

defining an element of $\pi_j(X, x_0)$. We have an induced map

$$I^j \times I \xrightarrow{j \times \mathrm{id}_I} X \times I$$

so quotienting $X \times I$ to the snowcone, and then to the reduced suspension, we have an induced map

$$(I^{j+1}, \partial I^{j+1}, J) \to (C'X, X \times \{1\}, \{*\}) \to (\Sigma X, \{*\}, \{*\}).$$

We call this composition q.

Show that the following diagram of groups commutes:

$$\begin{array}{ccc} \pi_{j+1}(C'X, X \times \{1\}, \{*\}) & & \\ \downarrow \partial & \overset{q_*}{\searrow} & \\ \pi_j(X, x_0) & \xrightarrow{\Sigma} & \pi_{j+1}(\Sigma X, x_0) \end{array}$$

Here, ∂ is the boundary map in the LES of relative homotopy groups. Σ is the map from part (c).

4. The Freudenthal Suspension Theorem

Assume that X is a CW complex with basepoint $x_0 \in X^0$.

(a) Using the homotopy excision theorem, prove:

THEOREM 4.1 (Freudenthal Suspension Theorem). If $\pi_j(X, x_0) = 0$ for all $j \leq n$, then the map

$$\pi_j(X, x_0) \to \pi_{j+1}(\Sigma X, x_0)$$

is an isomorphism for $j < 2n + 1$, and a surjection for $j = 2n + 1$.

(Hint: You can give ΣX a CW structure, and ΣX is the union of two sub-CW complexes.)

(b) Assume that $\pi_3(S^3) \cong \mathbb{Z}$. Compute $\pi_n(S^n)$ for all n.

(c) Assume that $\pi_1(S^1) \cong \mathbb{Z}$. Show that $\pi_n(S^n)$ is a cyclic group (i.e., a group with a single generator) for all n.

(d) For every $k \in \mathbb{Z}$, exhibit a pointed map from S^n to itself with degree k. (Recall that degree is defined via homology groups.)

(e) Assume that $\pi_1(S^1) \cong \mathbb{Z}$. Show that $\pi_2(S^2)$ and $\pi_3(S^3)$ are isomorphic to $\mathbb{Z}$.

5. Stable homotopy groups of spheres

(a) Fix an integer $j \geq 0$. Note that suspension defines a sequence of maps
$$\pi_j(S^0) \to \pi_{j+1}(S^1) \to \ldots \to \pi_{j+k}(S^k) \to \pi_{j+k+1}(S^{k+1}) \to \ldots.$$
Prove that there exists some $K \in \mathbb{Z}$ such that whenever $k \geq K$, the map
$$\pi_{j+k}(S^k) \to \pi_{j+k+1}(S^{k+1})$$
is a group isomorphism.

The group
$$\pi_j(\mathbb{S}) := \pi_{j+k}(S^k), \qquad k \geq K$$
is called the *jth stable homotopy group of spheres.*

If you can compute all of these, you will most likely be awarded the Fields medal. (Or at least get tenure at MIT or Harvard.)

(b) Compute $\pi_0(\mathbb{S})$.

(c) The definition for $\pi_j(\mathbb{S})$ also makes sense for negative j. That is, there is still a sequence of maps
$$\pi_{j+k}(S^k) \to \pi_{j+k+1}(S^{k+1}) \to \ldots$$
for $k \geq j$. Compute $\pi_j(\mathbb{S})$ for all negative j.

6. Quotients via mapping cones

Let $f : A \to X$ be a continuous map. It makes sense to talk about the space "X/A" by quotienting by the image of f, but it is often nicer to take M_f (which, you should recall, is homotopy equivalent to X), and to quotient out M_f by A.

Definition 6. *The space*
$$C_f := M_f/A \times \{0\}$$
is called the mapping cone *of f.*

(a) Let (C, Q) be a CW pair. If Q is contractible, show that the space C/Q is homotopy equivalent to C. (Hint: Produce a third space which is homotopy equivalent to both C/Q and to C. You'll have to use the one homotopical property that you know CW pairs have.)

(b) Let C_f be the mapping cone, and let $Q \subset C_f$ be the image of $A \times I$ under the composition
$$X \times I \to (A \times I) \coprod X \to M_f \to C_f.$$
If f is an inclusion of spaces, show that Q is homeomorphic to the cone on A.

(c) If f is an inclusion of CW complexes, show that (C_f, Q) can be given the structure of a CW pair.

(d) Show there is a homeomorphism

$$C_f/Q \cong X/A$$

when $f : A \to X$ is the inclusion of a subspace.

(e) Assume $f : A \to X$ is an inclusion of a sub-CW complex. Write a map

$$C_f \to X/A$$

and prove it's a homotopy equivalence.

7. Homotopy groups of quotients

In this problem, assume that (X, A) is an m-connected CW pair. Also assume that A is n-connected. (In particular, A is non-empty.)

Let

$$f : A \to X$$

be the inclusion map.

(a) Exhibit an isomorphism of homotopy groups

$$\pi_j(C_f, *) \to \pi_j(C_f, Q, *)$$

for all $j \geq 0$, where C_f and Q are spaces as in the previous problem.

(b) Exhibit an isomorphism of homotopy groups

$$\pi_j(C_f, *) \to \pi_j(X/A)$$

(c) Prove that if A is n-connected, then the pair (Q, A) is $(n + 1)$-connected.

(d) Construct a map

$$\pi_j(X, A, *) \to \pi_j(X/A, *)$$

and prove that it is an isomorphism for all $j \leq n + m$ and a surjection for $j = n + m + 1$.