Homework 7: Some basic computations

Notation and Remarks

This is a short homework meant to make sure we can all do some computations.

1. Degree and $\mathbb{R}P^n$

(a) Let $f, g : S^n \to S^n$ be two continuous maps. Show that $\deg(g \circ f) = \deg g \times \deg f$.

(b) Using cellular homology, and the CW structure we defined in class a few weeks ago, compute $H_*(\mathbb{R}P^n)$ for all n, including $n = \infty$. (Refer to a past homework for the degree of the antipodal map.)

(c) Show that the quotient map $S^n \to \mathbb{R}P^n$ is a covering map with fiber equal to a discrete space with 2 points.

(d) Compute $\pi_m(\mathbb{R}P^n)$ in terms of $\pi_m S^n$ for all $m \geq 2$. Also compute $\pi_1(\mathbb{R}P^n)$.

2. Homology is not a complete invariant

Exhibit two spaces with isomorphic homology groups that are not homotopy equivalent.

3. Relative homology is not always the reduced homology of the quotient

Let $X = [0, 1]$ and let

$$A = \{0\} \cup \{1/n\}_{n \in \mathbb{Z}_{\geq 1}}.$$

(a) Show that $H_1(X, A)$ is isomorphic to to a direct *sum* $\oplus_{n \in \mathbb{Z}_{\geq 0}} \mathbb{Z}$ of countably many copies of $\mathbb{Z}$.

(b) Show that the direct sum of countably many copies of $\mathbb{Z}$ is countable (as a set).

(c) Exhibit a surjection from $\tilde{H}_1(X/A)$ to the direct *product* $\prod_{n \in \mathbb{Z}_{\geq 0}} \mathbb{Z}$ of countably many copies of $\mathbb{Z}$. (Recall that the distinction between direct product and direct sum is that the sum can have only finitely many non-zero factors.)

(d) Show that the direct product of countably many copies of $\mathbb{Z}$ is uncountable as a set.

(e) State why $\tilde{H}_1(X/A)$ and $H_1(X, A)$ cannot be isomorphic groups.

(f) But we know that under many circumstances, we have an isomorphism $H_\bullet(X, A) \cong \tilde{H}_\bullet(X/A)$. What hypotheses are not met in this problem?

4. Homotopy groups of products

Let (X, x_0) and (Y, y_0) be pointed spaces. Show that there is an isomorphism

$$\pi_n(X \times Y, (x_0, y_0)) \cong \pi_n(X, x_0) \times \pi_n(Y, y_0)$$

for all $n \geq 0$. More generally, let $\{(X_\alpha, x_\alpha)\}_{\alpha \in \mathcal{A}}$ be a collection of pointed space. $\mathcal{A}$ may be a ridiculous indexing set (uncountable, even). Show

$$\pi_n(\prod_\alpha X_\alpha, (x_\alpha)) \cong \prod_\alpha \pi_n(X_\alpha, x_\alpha).$$

5. Excision applied to smash products

Show that if (X, x_0) is m-connected and (Y, y_0) is n-connected, then $X \wedge Y$ is $(m + n + 1)$-connected.

6. The Klein Bottle

(a) Find the universal cover of the Klein bottle.

(b) Compute π_m of the Klein bottle for all $m \geq 1$.