Homework 8

Due date: Friday, November 1.

0. Notation and Remarks

Pointed maps. Given any two pointed spaces $X = (X, x_0)$ and $Y = (Y, y_0)$ let

$$\mathrm{Maps}_*(X, Y)$$

denote the set of pointed, continuous maps. These are maps $f : X \to Y$ such that $f(x_0) = y_0$. Note that this mapping space is pointed—the constant map is the basepoint. We denote by

$$[X, Y] = \mathrm{Maps}_*(X, Y)/\sim$$

the *set* of homotopy classes of maps.

Mapping Spaces. Let X and Y be topological spaces. Recall that $\mathrm{Maps}(X, Y)$ is the space of continuous maps from X to Y, obtained by turning the compact-open topology into a compactly closed topology.

1. Topology of mapping spaces

(a) Let X and Y be compactly generated spaces. Prove that $\mathrm{Maps}(X, Y)$ is compactly generated. Can you weaken the assumptions on X and Y?

(b) Let X and Y be compactly generated spaces. Prove that two maps $f_0, f_1 : X \to Y$ are homotopic if and only if there is a continuous map

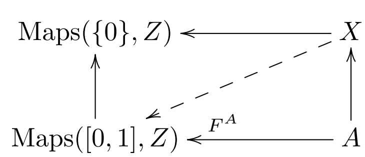
$$[0, 1] \to \mathrm{Maps}(X, Y)$$

such that $0 \mapsto f_0$ and $1 \mapsto f_1$.

It is a theorem of Milnor that $\mathrm{Maps}(X, Y)$ is homotopy equivalent to a CW complex if X and Y are CW complexes.

2. Cofibrations

A continuous map $i : A \to X$ is called a *cofibration* if for any space Z, the following dotted arrow exists, given that the rest of the diagram commutes:



This means that any homotopy of A extends to a homotopy on all of X.

(a) Let $f : X \to Y$ be any continuous map. Show that

$$X \to M_f, \qquad x \mapsto (x, 0)$$

is a cofibration. Recall that M_f is the mapping cylinder, defined to be

$$(X \times [0,1] \coprod Y)/(x, 1) \sim f(x).$$

(As a hint: Try using the technique from a previous homework, which shows that a certain subspace is a (not necessary deformation) retract of a product space).

(b) Show that any continuous map $f : X \to Y$ factors as a cofibration, followed by a homotopy equivalence:

$$X \hookrightarrow M \simeq Y$$

(c) State in one sentence, using previous homework, why the continuous map $X^k \to X$ of including the k-skeleton into a CW complex is a cofibration.

(d) Let $f : X \to Y$ be any continuous map between pointed spaces. The *homotopy cofiber of f* is defined to be the colimit (in the category of spaces) of the diagram

$$\begin{array}{ccc} X & \longrightarrow & M_f \\ \downarrow & & \\ * & & \end{array} .$$

If $f : X \to *$ is the constant map, find the homotopy cofiber of f. It is a space you've studied before.

3. Fibrations

A continuous map $p : E \to B$ is called a *fibration* if for any space Y, the following dotted arrow exists, given that the rest of the diagram commutes:

$$\begin{array}{ccc} Z & \longrightarrow & E \\ \downarrow{\scriptstyle (y,0)} & \nearrow & \downarrow{\scriptstyle p} \\ Z \times [0,1] & \longrightarrow & B \end{array}$$

This means that any homotopy of Z occurring in the base B can be lifted to the space E.

(a) Let $f : X \to Y$ be any continuous map. Let

$$N_f := X \times_f \text{Maps}([0,1], Y) = \{(x, \gamma) \text{ such that } \gamma(0) = f(x)\}$$

be the *mapping path space* of f. As a set, it is as indicated. As a space, we take the subspace topology inherited from $X \times \mathrm{Maps}([0,1], Y)$, then we apply the functor k to make it a compactly generated space.

Show that the map

$$N_f \to Y, \qquad (x, \gamma) \mapsto \gamma(1)$$

is a fibration.

(b) Show that N_f is homotopy equivalent to X.

(c) Show that any continuous map $f : X \to Y$ factors as a homotopy equivalence, followed by a fibration:

$$X \simeq N \to Y.$$

(d) Fix a map $f : X \to Y$ of pointed spaces. The *homotopy fiber of f* is the limit (in the category of spaces) of the following diagram:

$$\begin{array}{ccc} & & N_f \\ & & \downarrow \\ * & \longrightarrow & Y \end{array}$$

If $f : * \to Y$ is the map to the base point of Y, find the homotopy fiber of f. You may see this space again during this homework.

4. Based loop spaces and suspension-loop adjunction

Define the *based loop space of X* to be the topological space

$$\Omega X := \mathrm{Maps}_*(S^1, X).$$

Recall that subspaces of a compactly generated space may not be compactly generated, so by the above, we mean the k-ification of the subspace. It's a theorem of Milnor that if X is homotopy equivalent to a CW complex, so is ΩX. Note that ΩX has a natural basepoint, called the constant loop at x_0.

You may take for granted that Ω defines a functor from pointed spaces to pointed spaces. On maps, a map $f : X' \to X$ is sent to map $\Omega X' \to \Omega X$ in the obvious way: post-composing a loop into X' by f.

(a) Let X, Y, Z be compactly generated spaces. Prove there is a homeomorphism of spaces

$$\mathrm{Maps}_*(X \wedge Y, Z) \cong \mathrm{Maps}_*(X, \mathrm{Maps}_*(Y, Z)).$$

(b) Let X and Z be compactly generated. Show there is an isomorphism of sets

$$[\Sigma X, Z] \cong [X, \Omega Z]$$

where ΣX is the reduced suspension.

5. Homotopy Groups

(a) Let $\Omega^n Y = \Omega \ldots \Omega Y$, where Ω is performed n times. Show that for all n, there is an isomorphism of sets

$$\pi_0(\Omega^n Y) \cong \pi_n Y.$$

(b) Show that for all n, there is an isomorphism of sets

$$[\Sigma^n(S^0), Y] \cong \pi_n Y.$$

(c) Show that if Y is a compactly generated, connected space, then you can recover all the homotopy groups of Y by knowing all the homotopy groups of ΩY, and vice versa. There's a subtlety involved for $\pi_0 \Omega Y$. You can simply state that there's a bijection of sets in this case, if you like.

6. Covering Spaces

A map $p : E \to B$ is called a *covering*, or *a covering space* if it is a surjection, and if for each $b \in B$, there is an open neighborhood $b \in U_b \subset B$ such that each connected component of $p^{-1}(U_b)$ is open in E, and is mapped *homeomorphically* onto U_b by p.

Example. Let $p : \mathbb{R} \to S^1$ be the map sending $t \mapsto e^{i2\pi t}$. Then any connected, open arc U on S^1 has preimage $p^{-1}(U)$ consisting of a disjoint union of connected open intervals, each of which is homeomorphic to U under the projection map.

This *local homeomorphism* property makes the world go round when we're dealing with covering spaces, as we'll see.

(a) Let $p : E \to B$ be a covering space. For any $x \in E, b = p(x)$, show that any path $\gamma : [0, 1] \to B$ with $\gamma(0) = p(x)$ has a unique lift to E. (Hint: We did a "non-unique" version of this in class for vibrations.)

(b) Let $p : E \to B$ be a covering space with B path-connected. Show that p is a Serre fibration. (Hint: Is p a fiber bundle?)

(c) If $p : E \to B$ is a covering space, show that the lifting (i.e., the dotted arrow) guaranteed by the homotopy lifting property must be unique.

7. An application of your solutions to the covering space problem

(a) Let $p : E \to B$ be a covering space. Assume E and B are path-connected and locally contractible. Show that there is an isomorphism

$$\pi_n(E) \cong \pi_n(B)$$

for all $n \geq 2$.

(b) Given that the real line is a covering space of S^1, compute $\pi_n(S^1)$ for all $n \geq 2$.

(c) Show that if $p_1 : E_1 \to B_1$ is a covering space, and $p_2 : E_2 \to B_2$ is a covering space, then the map

$$E_1 \times E_2 \to B_1 \times B_2, \qquad (e_1, e_2) \mapsto (p_1 e_1, p_2 e_2)$$

is a covering space.

(d) Show that for all $k \geq 0$, the k-torus $T^k := S^1 \times \dots S^1$ has vanishing higher homotopy groups. That is,

$$\pi_n(T^k) = 0 \qquad \text{whenever } n \geq 2.$$

(e) More generally, if any space X admits a covering space $E \to X$ where E is contractible, show that

$$\pi_n(X) = 0 \qquad \text{for all } n \geq 2.$$