Homework 9: Hurewicz Theorem, and toward Cohomology

Due date: Friday, November 8.

0. Notation and Remarks and Goals

In this homework, we'll do some simple applications of Hurewicz, prove some simple properties of (co)fibrations and how they behave with respect to some easy (co)limits, and do some homological algebra. (I.e., play with chain complexes.)

1. Simply connected spaces are easier

Let X and Y be pointed CW complexes such that $\pi_k(X) = \pi_k(Y) = 0$ for $k = 0, 1$. Assume that $f : X \to Y$ is a map of pointed spaces inducing an isomorphism on homology groups. Show that f is a homotopy equivalence. (Hint: Replace f by a cofibration; this gives you a long exact sequence of homotopy groups, and of homology groups. What does the relative version of the Hurewicz theorem tell you?)

2. A fun problem

(a) True or false: $S^2 \vee S^4$ and $\mathbb{C}P^2$ are not homotopy equivalent. (This will be easier with cohomology, which we'll learn next week. But you have the tools to answer this question right now!)

(b) True or false: There exists no map from S^4 to $\mathbb{C}P^2$ inducing an isomorphism on H_4.

3. Pushouts preserve cofibrations

Let $f : A \to Y$ and $g : A \to X$ be two continuous maps. The *pushout* of the diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d] & Y \\ X & \end{tikzcd}$$

is defined to be colimit of the diagram.

(a) Show that the topological space

$$X \cup_A Y := (X \coprod Y \coprod A)/a \sim f(a), a \sim g(a),$$

(together with the obvious map from the diagram) is a pushout of the above diagram.

(b) If A is a point, show that the pushout is homeomorphic to the wedge sum.

(c) Show that if $A \to Y$ is a cofibration, then the pushout map $X \to X \cup_A Y$ is a cofibration.

4. Pullbacks preserve fibrations

Let $p_Y : Y \to B$ and $p_X : X \to B$ be two continuous maps. The *pullback* of the diagram

$$\begin{tikzcd} & X \arrow[d, "p_X"] \\ Y \arrow[r, "p_Y"] & B \end{tikzcd}$$

is defined to be the limit of the diagram.

(a) Show that the topological space

$$X \times_B Y := \{(x, y) \text{ such that } p_X(x) = p_Y(y) \in B\} \subset X \times Y$$

(together with the obvious map to the diagram) is a pullback of the above diagram.

(b) If $B = *$ is a point, show that the pullback is homeomorphic to the direct product.

(c) Show that if $X \to B$ is a Serre fibration, then the map $X \times_B Y \to Y$ is a Serre fibration.

5. Duality of fibrations and cofibrations

(a) Let $i : A \to X$ be a cofibration. Let Q be any space. Show that the map

$$i \times \mathrm{id}_Q : A \times Q \to X \times Q$$

is a cofibration.

(b) Let $i : A \to X$ be a continuous map. Let B be a space. Consider the "composition map"

$$\mathrm{Maps}(X, B) \to \mathrm{Maps}(A, B) \qquad g \mapsto g \circ i.$$

If i is a cofibration, show that this map is a fibration.

6. Tensor products of (co)chain complexes.

Fix a commutative ring R.

By a *cochain complex A over R*, we mean a sequence of R-modules

$$\ldots \to A^{-1} \to A^0 \to A^1 \to A^2 \to \ldots$$

together with maps $d^i : A^i \to A^{i+1}$, such that

$$d^{i+1} \circ d^i = 0.$$

Note we use superscripts, and note that the differentials *increase* the indexing. We have seen the case of $R = \mathbb{Z}$ before, for chain complexes (as opposed to cochain complexes.)

We define the *nth cohomology group* to be the R-module

$$H^n(A) := \ker(d^n)/\operatorname{image}(d^{n-1}).$$

(a) Let $f = (f^n : A^n \to B^n)$ be a sequence of R-modules map. Then f is called a *cochain map*, or sometimes just a *chain map*, if

$$d_B^n f^n = f^{n+1} d_A^n,$$

or more succinctly, $df = fd$. We will write $f : A \to B$ to mean (f^n), for brevity. Show that a composition of two chain maps is again a chain map.

(b) Let A and B be two cochain complexes of R-modules. Define their tensor product

$$A \otimes_R B$$

to be the following cochain complex: The nth R-module is given by

$$(A \otimes_R B)^n := \bigoplus_{i+j=n} A^i \otimes_R B^j$$

and the differential is given by

$$d(a \otimes b) := da \otimes b + (-1)^i a \otimes db \qquad \text{for all } a \in A^i, b \in B^j$$

and extended linearly. To be explicit, for a general linear combination of terms in $(A \otimes_R B)^n$, we have

$$d(\sum_{i,j} a_i \otimes b_j) = \sum_{i,j} (da_i \otimes b_j + (-1)^{|a_i|} a_i \otimes db_j).$$

Where $|a_i|$ is the degree of a_i; it's the number k for which $a_i \in A^k$.

Verify that $A \otimes_R B$ is a cochain complex.

(c) Let $R = \mathbb{Z}$. Let B be the cochain complex with $B^0 = \mathbb{Z}/2\mathbb{Z}$ and $B^j = 0$ for $j \neq 0$. Let A be the cochain complex

$$\ldots \longrightarrow 0 \longrightarrow \mathbb{Z} \xrightarrow{\times 2} \mathbb{Z} \longrightarrow 0 \longrightarrow \ldots, \qquad A^0 = A^1 = \mathbb{Z}.$$

Show that

$$H^\bullet(A \otimes_{\mathbb{Z}} B) \not\cong H^\bullet(A) \otimes_{\mathbb{Z}} B^0$$

where the rigthand side is just the usual tensor product of groups, while the lefthand side uses the tensor product of cochain complexes.

(d) Given two cochain complexes A and B of R-modules, use the techniques from Homework One to define a cochain complex
$$\hom_R^\bullet(A, B).$$

(e) Verify that for three R-modules A, B and C, we have an isomorphism of R-modules
$$\hom(A \otimes_R B, C) \cong \hom(A, \hom(B, C))$$
where hom is the R-module of R-module morphisms.

(f) Verify that for three cochain complexes of R-modules A, B and C, we have an isomorphism of cochain complexes (of R-modules)
$$\hom^\bullet(A \otimes_R B, C) \cong \hom^\bullet(A, \hom^\bullet(B, C))$$
where $\hom^\bullet$ is the cochain complex you defined above.