Homework 10

Due date: Friday, November 15.
Hand in every problem ***except problems 6(a) and 7(c), 7(d)***.

0. Notation and Remarks

Notation. As usual, $H_n(X)$ will denote homology with coefficients in $\mathbb{Z}$. $H_n(X;\pi)$ will denote homology with coefficients in an abelian group π.

$H^n(X)$ will denote cohomology with coefficients in $\mathbb{Z}$. $H^n(X;\pi)$ will denote cohomology with coefficients in an abelian group π.

Goals. In this problem set you'll (repeatedly) use the Künneth formula and the universal coefficient theorem to compute homology with different coefficients, and cohomology with different coefficients. You'll also see via example that the splittings in these theorems cannot be natural.

Finally, there is also a problem about Eilenberg-Maclane spaces. It turns out that for any abelian group G and any integer $n \geq 0$, there is a canonical isomorphism

$$H^n(X;G) \cong \pi_0 \operatorname{Maps}(X, K(G,n))$$

whenever X is a compactly generated, weak Hausdorff space. It is then natural to ask: Where does the addition on π_0 of this mapping space come from? (This is hinted at in one of the problems.) Is it possible to derive the long exact sequence of relative cohomology groups from this perspective? And how about the multiplication on cohomology?

This "mapping space" perspective on cohomology may or may not be covered during the week of my absence. Nor is it needed for this problem set. But I wanted to expose you to this perspective on cohomology, at least in passing. (Which I just did!)

1. Additivity of Ext

For any choice of abelian groups A, A', and B, prove

$$\operatorname{Ext}^1(A \oplus A' B) \cong \operatorname{Ext}^1(A,B) \oplus \operatorname{Ext}^1(A',B).$$

You may assume Ext^1 is independent of the choice of free resolution.

2. Cohomology of $\mathbb{C}P^\infty$ and spheres

(a) Assume that the homology groups of X are free in each degree, and finitely generated in each degree. Prove that $H^n(X) \cong \hom(H_n(X),\mathbb{Z}) \cong H_n(X)$ for all n.

(b) Compute the cohomology groups $H^n(\mathbb{C}P^\infty)$ for all n.

(c) Compute the cohomology groups $H^n(S^k)$ for all n, k.

3. Moore spaces and non-naturality of splittings

Let G be an abelian group. Let $M(G, n)$ be a space whose reduced homology groups are equal to G in degree n, but equal to 0 in all other degrees. $M(G, n)$ is called a *Moore space.*

(a) Construct a CW complex $M(\mathbb{Z}/m\mathbb{Z}, n)$ having only three cells: One in degrees 0, n, and $n+1$.

(b) By considering the map
$$g : M(\mathbb{Z}/m\mathbb{Z}, n) \to S^{n+1}$$
collapsing the n-skeleton, prove that the splitting in the universal coefficient theorem cannot be natural. (What does g^* do on cohomology?)

(c) By considering the map
$$g \times \mathrm{id} : M(\mathbb{Z}/m\mathbb{Z}, n) \times M(\mathbb{Z}/m\mathbb{Z}, n) \to S^{n+1} \times M(\mathbb{Z}/m\mathbb{Z}, n)$$
prove that the splitting in the Künneth theorem cannot be natural.

4. Homology of $\mathbb{R}P^\infty$.

(a) Compute $H_k(\mathbb{R}P^\infty; \mathbb{Z}/2\mathbb{Z})$ for all k.

(b) Compute $H_k(\mathbb{R}P^\infty; \mathbb{Z}/m\mathbb{Z})$ for all k, and for any odd integer $m \geq 3$.

(c) Compute $H_k(\mathbb{R}P^\infty \times \mathbb{R}P^\infty)$ for all k.

(d) Compute $H_k(\mathbb{R}P^\infty \times \mathbb{R}P^\infty; \mathbb{Z}/2\mathbb{Z})$ for all k.

5. Cohomology of $\mathbb{R}P^\infty$.

(a) Compute $H^k(\mathbb{R}P^\infty)$ for all k.

(b) Compute $H^k(\mathbb{R}P^\infty; \mathbb{Z}/2\mathbb{Z})$ for all k.

6. Eilenberg-MacLane spaces

Fix an abelian group G. A connected CW complex space with $\pi_n \cong G$, with $\pi_k \cong 0$ for all $k \neq n$, is called an *Eilenberg-MacLane space*, and written $K(G, n)$.

(a) Fix a group G and an integer $n \geq 1$. Using the construction in the proof of the CW approximation theorem (and Whitehead's theorem), prove that two CW complexes that are $K(G, n)$ must be homotopy equivalent. (This problem is hard.)

(b) Let X be a CW complex such that X is a $K(G, n)$ with G abelian. Prove that for any integer $m \geq 1$, there exists a space Y such that X is homotopy equivalent to an m-fold loop space $\Omega^m Y$.

You may assume that if Z is homotopy equivalent to some CW complex, then ΩZ is homotopy equivalent to a CW complex as well. (This is a theorem of Milnor.) You may also assume part (a) of this exercise.

7. Cohomology of surfaces

(a) Compute the cohomology groups of the torus $T^2 = S^1 \times S^1$.

(b) Prove that as a graded ring, $H^*(T^2)$ is isomorphic to

$$\mathbb{Z}[\alpha, \beta]/(\alpha^2, \beta^2, \alpha\beta + \beta\alpha)$$

where α and β are in degree 1.

(c) Compute the cohomology groups of the surface of genus g.

(d) Note that the surfaces of genus g admits a collapse map to the wedge of g tori. Prove that you can find a basis $\{\alpha_1, \beta_1, \ldots, \alpha_g, \beta_g\}$ for H^1 such that the following relations hold in the ring H^*:
  (a) $\alpha_i^2 = 0$
  (b) $\beta_i^2 = 0$
  (c) $\alpha_i\beta_i + \beta_i\alpha_i = 0$
  (d) $\alpha_i\beta_{j\neq i} = 0$
  (e) $\alpha_i\beta_i - \alpha_j\beta_j = 0$