Homework 11

Due date: Friday, November 22. **Hand in every problem except problem 5.**

Notation and Remarks. We let Pairs denote the category of pairs of spaces, with continuous maps of pairs between them. To be explicit, objects are pairs $A \subset X$, and morphisms $f : (X, A) \to (Y, B)$ are continuous maps $f : X \to Y$ such that $f(A) \subset B$.

As usual, a homotopy between such maps is a homotopy of $F : X \times [0, 1] \to Y$ such that $F(A \times [0, 1]) \subset B$.

Axioms for cohomology. Recall from class the Eilenberg-Steenrod axioms for cohomology. You'll need this in Problem 5.

Definition 1. *A cohomology theory is a sequence of functors*

$$K^n : \mathsf{Pairs}^{\mathrm{op}} \to \mathsf{Groups}, \qquad n \in \mathbb{Z}$$

together with natural transformations

$$\delta : K^{n-1}(A, \emptyset) \to K^n(X, A), \qquad n \in \mathbb{Z}$$

such that the following properties hold:

(1) *(Homotopy.) If $f, g : (X, A) \to (Y, B)$ are homotopic as maps of pairs, then $K^n(f) = K^n(g) : K^n(Y, B) \to K^n(X, A)$ for all n.*
(2) *(Excision.) If $A \subset X$ has closure contained in an open set U, then the inclusion $(X - A, U - A) \to (X, U)$ induces an isomorphism*

$$K^n(X, U) \to K^n(X - A, U - A). \qquad n \in \mathbb{Z}$$

(3) *(Additivity.) If $X = \coprod_\alpha X_\alpha$ and $A = \coprod_\alpha A_\alpha$ with (X_α, A_α) pairs of spaces, then the maps $(X_\alpha, A_\alpha) \to (X, A)$ induce an isomorphism*

$$K^n(X, A) \to \prod_\alpha K^n(X_\alpha, A_\alpha) \qquad n \in \mathbb{Z}$$

(4) *(Exactness.) The sequence*

$$\cdots \longrightarrow K^n(X, A) \longrightarrow K^n(X, \emptyset) \longrightarrow K^n(A, \emptyset) \xrightarrow{\delta} K^{n+1}(X, A)$$

is exact.

Remark 0.1. Note that $K^n(pt, \emptyset)$ need not be 0 for $n \neq 0$. This is a very general definition.

Remark 0.2. Note that by excision, and the usual commutative diagram of pairs of spaces

$$\begin{array}{ccccc} (X,A) & \longrightarrow & (X,U) & \longleftarrow & (X-A, U-A) \\ \downarrow & & \downarrow & & \downarrow \\ (X/A, A/A) & \longrightarrow & (X/A, U/A) & \longleftarrow & (X/A - A/A, U/A - A/A) \end{array}$$

we can bootstrap the far-right vertical homeomorphism to conclude that the leftmost vertical map induces an isomorphism $K^n(X/A, A/A) \to K^n(X, A)$.

1. Cohomology and direct product rings

Let X_α be a collection of pointed spaces and let $X = \vee_\alpha X_\alpha$. Prove there is an isomorphism of (non-unital) graded rings

$$\tilde{H}^*(X) \cong \prod_\alpha \tilde{H}^*(X_\alpha).$$

(By definition, if K and H are graded rings, the nth graded piece of $K \times H$ is $K_n \times H_n$.)

2. Relative cup product via cross product

Given two spaces X and Y, let $p_1 : X \times Y \to X$ and $p_2 : X \times Y \to Y$ denote the two projections. Fix $A \subset X$ and $B \subset Y$.

Recall we have a cross product map

$$\times : H^*(X, A) \otimes H^*(Y, B) \to H^*(X \times Y, A \times Y + X \times B) \cong H^*(X \times Y, A \times Y \cup X \times B)$$

defined by the map

$$\alpha \otimes \beta \mapsto p_1^*(\alpha) \cup p_2^*(\beta).$$

On the other hand, the cup product for X restricts to the relative cup product

$$\cup : H^*(X, A) \otimes H^*(X, A) \to H^*(X, A), \qquad \alpha \otimes \beta \mapsto \alpha \cup \beta$$

Using the cross product for $X = Y$, $A = B$, prove that the diagram

$$\begin{array}{ccc} H^*(X,A) \otimes H^*(X,A) & \xrightarrow{\cup} & H^*(X,A) \\ \downarrow{\scriptstyle p_1^* \otimes p_2^*} & & \uparrow{\scriptstyle \Delta^*} \\ H^*(X \times X, A \times X) \otimes H^*(X \times X, A \times X) & \xrightarrow{\cup} & H^*(X \times X, A \times X \cup X \times A) \end{array}$$

commutes.

(This is the relative version of the statement from class that

$$\begin{array}{ccc} H^*(X) \otimes H^*(X) & \xrightarrow{\cup} & H^*(X) \\ \downarrow{\scriptstyle p_1^* \otimes p_2^*} & & \uparrow{\scriptstyle \Delta^*} \\ H^*(X \times X) \otimes H^*(X \times X) & \xrightarrow{\cup} & H^*(X \times X) \end{array}$$

commutes.)

3. Cup products of suspensions vanish

Assume $X = A \cup B$ where A and B are contractible open subsets with $A \cap B \neq \emptyset$.

(a) Using the relative cup product map

$$H^k(X, A) \otimes H^l(X, B) \to H^{k+l}(X, A \cup B)$$

prove that for any k, l such that $k + l \geq 1$, the map

$$\cup : H^k(X) \otimes H^l(X) \to H^{k+l}(X)$$

is zero. (Hint: You may want to use the previous problem.)

(b) Let X be any space. Prove that the cup product map on ΣX is always zero for elements in degree k, l with $k + 1 \geq 1$.

4. Weak homotopy equivalences preserve cohomology rings

Let X and Y be pointed spaces, and let $f : X \to Y$ be a weak homotopy equivalence. Prove that it induces an isomorphism of graded rings $f^* : H^*(Y) \to H^*(X)$.

5. The maps in the Universal Coefficient Theorem and the Künneth Formula

Assume that $A_\bullet$ is a chain complex such that A_n is a free abelian group for all n. (This is nice because any subgroup of a free abelian group is once again free.) Let B be any other chain complex. The following problems are not easy.

(a) Verify that the map

$$H^n(\hom(A_\bullet, \mathbb{Z})) \to \hom(H_n(A_\bullet), \mathbb{Z}), \qquad [f] \mapsto ([a] \mapsto f(a))$$

is a surjection.

(b) Verify that the map

$$\bigoplus_{p+q=n} H_p(A_\bullet) \otimes H_q(B_\bullet) \to H_n(A_\bullet \otimes B_\bullet), \qquad [a] \otimes [b] \mapsto [a \otimes b]$$

is an injection.

6. Axioms for cohomology, Part I

Suppose L and K are cohomology theories for CW pairs. Let $u : L \to K = \{u^n : L^n \to K^n\}$ be a natural transformation between them. The goal of the next sequence of problems is to begin proving:

Theorem 6.1. If u induces an isomorphism $H^n(pt, \emptyset) \cong K^n(pt, \emptyset)$ for all n, then $u : H^n(X, A) \to K^n(X, A)$ is an isomorphism for all CW pairs (X, A).

In the following problem you may assume

Lemma 6.2 (The Five Lemma). If

$$\begin{array}{ccccccccc} Q & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & Z \\ \downarrow \cong & & \downarrow \cong & & \downarrow & & \downarrow \cong & & \downarrow \cong \\ Q' & \longrightarrow & A' & \longrightarrow & B' & \longrightarrow & C' & \longrightarrow & Z' \end{array}$$

is a commutative diagram of exact sequences, and all the vertical maps are isomorphisms where indicated, then the middle vertical map is also an isomorphism.

(a) Prove using a long exact sequence that if

$$u^n : L^n(X, \emptyset) \to K^n(X, \emptyset)$$

is an isomorphism for all spaces X and all n, then u is an isomorphism for all pairs (X, A).

(b) Prove that if $u^n : L^n(pt, \emptyset) \to K^n(pt, \emptyset)$ is an isomorphism for all n, then it is an isomorphism for all zero-dimensional CW complexes.

(c) Assume we have proven that u^k is an isomorphism for all $(X, \emptyset)$ for $k < n$ and $X^{n-1} = X$. Prove using a long exact sequence that $u^n : L^n(D^n, \partial D^n) \to K^n(D^n, \partial D^n)$ is an isomorphism.

(d) Assume we have proven that u^k is an isomorphism for all $(X, \emptyset)$ for $k < n$ and $X^{n-1} = X$. Let $Y^n = Y$ be a CW complex of dimension n, and let

$$\Phi : (\coprod_\alpha D^n, \coprod_\alpha \partial D^n) \to (Y^n, Y^{n-1})$$

be the map induced by the attaching maps. Show that

$$K^m(\Phi) : K^m(Y^n, Y^{n-1}) \to K^m(\coprod D^n, \coprod \partial D^n), \qquad L^m(\Phi) : L^m(Y^n, Y^{n-1}) \to L^m(\coprod D^n, \coprod \partial D^n),$$

are isomorphisms for all m. (For instance, by passing to the pair $(\vee S^n, *)$.)

(e) Conclude that if $u^m : L^m(pt, \emptyset) \to K^m(pt, \emptyset)$ is an isomorphism for all m, then $u^m : L^m(X, \emptyset) \to K^m(X, \emptyset)$ is an isomorphism for all finite-dimensional CW complexes X, and for all m.

We will stop here, having only proven the theorem when X and A are both finite dimensional. The infinite-dimensional case requires some categorical bru-ha-ha that we haven't discussed.