Homework 12: Cohomology rings of some product spaces

Due Monday, December 2nd, 2013. Hand in every problem **except Problem 7(a).**

Throughout this homework, you may replace the ring R by the integers, since we haven't talked about cohomology with coefficients explicitly enough.

0. Goals

In this homework set you'll prove a cool theorem in a cool way.

THEOREM 0.1. Let X and Y be arbitrary spaces. Assume $H^k(Y;R)$ is a free, finitely generated R-module for all k. Then the cross product

$$H^*(X;R) \otimes_R H^*(Y;R) \to H^*(X \times Y;R)$$

is an isomorphism of graded rings.

This is cool because now you can compute cohomology *rings* for a lot of product spaces. It's proved in a cool way because you will see a wonderful principle in action: Cohomology theories are determined by their value on a point.

REMARK 0.2. If $R = \mathbb{Z}$, a finitely generated free R-module is simply a free abelian group on finitely many generators. Further, tensoring over $R = \mathbb{Z}$ is the usual tensor product of abelian groups, and $H^*(X;Z) = H^*(X)$.

1. h, k and the natural transformation

Fix a CW complex Y. We define two sequences of functors

$$h^n, k^n : \mathsf{Pairs} \to \mathsf{AbGrps}$$

as follows:

$$h^n(X,A) = \bigoplus_i H^i(X,A;R) \otimes_R H^{n-i}(Y;R)$$

$$k^n(X,A) = H^n(X \times Y, A \times Y; R).$$

Note h^n is the term that appears in the Kunneth formula.

On morphisms, any map $f : (X,A) \to (X',A')$ induces maps

$$h^n(f) = \bigoplus_i (f^* \otimes_R \mathrm{id}_{H^{n-i}(Y;R)}), \qquad k^n(f) = (f \times \mathrm{id}_Y)^*.$$

Where f^* denotes the map on cohomology induced by the continuous map f.

(a) Show that the cross product

$$\times : H^i(X, A; R) \otimes H^{n-i}(Y, \emptyset; R) \to H^n(X \times Y, A \times Y; R), \qquad [\phi] \otimes [\psi] \mapsto [p_1^*\phi \cup p_2^*\psi]$$

(really, the direct sum of it over all $0 \leq i \leq n$) defines a natural transformation u^n from h^n to k^n for every n. (Note that the Y variable is fixed!)

(b) Show that the cross product above is a map of graded rings. (Recall that given two graded commutative rings A and B, the product on $A \otimes B$ is defined by

$$(a_1 \otimes b_1)(a_2 \otimes b_2) := (-1)^{|a_2||b_1|} a_1 a_2 \otimes b_1 b_2.$$

We mentioned this briefly in class.)

In the problems that follow, remember that singular cohomology—i.e., H^* itself—satisfies the Eilenberg-Steenrod axioms.

2. Homotopy

Let f, g be continuous maps of pairs between (X, A) and (X', A'). Prove that if f and g are homotopic as maps of pairs, then $h^n(f) = h^n(g)$ for all n, and likewise for k^n.

3. Excision

Show that both h and k satisfy the excision axiom.

4. Exactness

In what follows, you'll want to bootstrap the usual long exact sequences of relative cohomology groups.

(a) Show that k satisfies the exactness axiom.

(b) Show that h satisfies the exactness axiom. (This is slightly more annoying.)

5. Products

(a) Show that k satisfies the product axiom.

(b) Show that if N is a finitely generated, free R module, then

$$(\prod_\alpha M_\alpha) \otimes_R N \cong \prod_\alpha (M_\alpha \otimes_R N).$$

Show that h satisfies the product axiom.

6. Compatibility with δ

Let $\delta : H^k(X, A; R) \to H^{k+1}(X, A; R)$ be the connecting homomorphism for the LES in relative cohomology. Finally, let $\partial : H_{n+1}(X, A) \to H_n(A)$ be the connecting map in the LES for relative homology (with $\mathbb{Z}$ coefficients).

(a) Let

$$h : H^{n+1}(X, A); R) \to \hom(H_{n+1}(X, A), R)$$

denote the map in the universal coefficient theorem. (Recall that the relative cochain complex is actually a hom complex!)

Prove that the diagram

$$\begin{array}{ccc} H^n(A; R) & \xrightarrow{\ \delta\ } & H^{n+1}(X, A; R) \\ \Big\downarrow h & & \Big\downarrow h \\ \hom(H_n(A), R) & \xrightarrow{\ -\circ\partial\ } & \hom(H_{n+1}(X, A), R). \end{array}$$

is commutative. (This is standard but tedious diagram chasing.)

(b) Prove that for any pair (X, A) and any space Y, the diagram

$$\begin{array}{ccc} H^k(A; R) \otimes_R H^l(Y; R) & \xrightarrow{\ \delta\otimes\mathrm{id}\ } & H^{k+1}(X, A; R) \otimes_R H^l(Y; R) \\ \Big\downarrow \times & & \Big\downarrow \times \\ H^{k+1}(A \times Y; R) & \xrightarrow{\ \delta\ } & H^{k+l+1}(X \times Y, A \times Y; R) \end{array}$$

commutes. Here, the vertical maps are the cross product.

7. Point

(a) Show that the natural transformations u^n induce isomorphisms on h^n and k^n for $(X, A) = (pt, \emptyset)$.

(b) Explain how you have proven the main theorem. (You may assume a theorem from previous homework.) Make sure you explain why the isomorphism of $h^n(X, A), k^n(X, A)$ as groups is enough to show we actually have an isomorphism of rings.

(c) Where did you have to use the hypothesis on Y?

8. Applying the theorem

(a) Phil will show you this week that $H^*(\mathbb{C}P^n)$ is actually isomorphic to the polynomial ring $\mathbb{Z}[x]/x^{n+1}$, with x in degree 2. Compute the cohomology ring of $\mathbb{C}P^n \times \mathbb{C}P^m$ for $n, m \geq 0$.

(b) Compute the cohomology ring of $S^n \times S^m$.

(c) Show $S^n \times S^m$ is not homotopy equivalent to $S^n \vee S^m \vee S^{n+m}$. Could you have shown this by simply knowing the groups H^* and H_*?