Bridgeland stability conditions

Lectures delivered by Hiro Tanaka
Notes by Akhil Mathew

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Contents

Introduction

Hiro Tanaka taught a course (Math) on spectra and stable homotopy theory at Harvard in Fall 2013. These are my "live-TEXed" notes from the course.

Conventions are as follows: Each lecture gets its own "chapter," and appears in the table of contents with the date.

Of course, these notes are not a faithful representation of the course, either in the mathematics itself or in the quotes, jokes, and philosophical musings; in particular, the errors are my fault. By the same token, any virtues in the notes are to be credited to the lecturer and not the scribe. Thanks to Emily Riehl and Arnav Tripathy for pointing out several mistakes.

Please email corrections to `amathew@college.harvard.edu`.

Lecture 1
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§1 Logistics

I'd like to start with some logistics. My email address is `hirohirohiro@gmail.com`; my office is 341, in the back of the library. Office hours for this particular class are by *lunch appointment.* My website has official office hours, but those are for my first-year graduate topology class.

A lot of this class will be seminar style, and 12 talks will be given by students, mostly on basic material. There are three teams:

1. Team Edward. (Weeks 4-5: category of coherent sheaves on a variety X, and properties as an abelian category.)

2. Team Jacob. (Weeks 5-6: basics of ∞-categories, stable ∞-categories.)

3. Team Apple. (Final two weeks: applications and conjectures related to the minimal model program, the reconstruction problem — reconstructing a variety from the derived category of coherent sheaves, possibly together with stability conditions, mirror symmetry/dynamical systems.)

See the iSite, linked to by `math.harvard.edu/~hirolee`. Please fill out the survey by 5pm tomorrow if you plan to attend the course twice. The iSite has a link to the syllabus of the talks.

§2 Notes

Every week, we should have two designated note-takers.

§3 Other things

There's a topic I've been thinking about this summer, on connections to K-theory and factorization homology. I'm giving a talk at the BU seminar tomorrow.

§4 Definitions

I'd like to start with an example that goes back at least to Mumford; Wikipedia tells me it goes back to Hilbert.

We're going to fix a smooth projective curve X, over $\mathbb{C}$. Fix a vector bundle $E \to X$. There are two natural topological invariants attached to E:

1. The rank $\mathrm{rk}E$ of E.

2. The degree $\deg E$ of E. If E is a line bundle, take some generic meromorphic section of E, and count the zeros and the poles of that section; subtract one from the other. In general, the degree is the degree of the top exterior power.

1.1 Definition. The **slope** of E is defined to be

$$\mu(E) \stackrel{\text{def}}{=} \frac{\deg E}{\text{rk} E}.$$

1.2 Definition. E is called **stable** if whenever $F \subsetneq E$ is a subbundle (nonzero), then

$$\mu(F) < \mu(E).$$

E is called **semistable** if the strict inequality is replaced by $\leq$ in the above.

Somehow, you can now look at the moduli space of, not all vector bundles on X, but of all semistable vector bundles on X (of fixed rank, degree). The theorem of Mumford is that if you look at all the stable vector bundles, it has a coarse moduli space given by a quasi-projective variety. The point is that this notion of stability or semistability gets you a nice moduli space.

If you were at the Gelfand conference, Kontsevich mentioned in his talk that there are different stability conditions to put on a category, and the space of such conditions is a complex manifolds. The topology that is really interesting, though, is the moduli space for a given stability condition. The topology changes as the stability condition changes, and that's one of the reasons people are interested in this.

Let's now talk about different stability conditions, and how one gets different moduli spaces. I have two numbers here, called rank and degree, and somehow I lost information by passing to a single number.

Remark.

1. First of all, why pass to quotients when we can remember the pair $(-\deg, \text{rk})$?
2. These make sense for coherent sheaves as well. The rank is the dimension of the stalk at the generic point. I don't know a satisfactory definition of the degree, but you define so that it has the right formal properties. (N.B. I'm thinking of a vector bundle as a sheaf of sections.)

1.3 Definition. Let $\mathcal{C}$ be the category of coherent sheaves on the curve X. We define a function

$$Z : \text{ob}\mathcal{C} \to \mathbb{C},$$

which has the definition

$$Z(E) = i\text{rk}(E) - \deg(E), \tag{1}$$

where $i = \sqrt{-1}$.

First, note that the rank is a nonnegative integer, so Z takes values in the upper half plane. Thus we can restrict the image. What are the sheaves of rank zero on a curve? The torsion sheaves, supported on finite sets. As a result, the image of Z is in the upper half plane minus the positive real axis.

Here are the basic properties.

1. $Z(E) \in \{x + iy | y \geq 0 \text{ or } y = 0 \text{ and } x \leq 0\}$.
2. If $Z(E) = 0$, then $E \simeq 0$. This is really important.

3. The category $\mathcal{C} = \mathrm{Coh}(X)$ is an abelian category; there's a definition of a short exact sequence. Given a short exact sequence

$$0 \to E_1 \to E \to E_2 \to 0,$$

in $\mathcal{C}$, then we know that rank and degree are additive. So

$$Z(E) = Z(E_1) + Z(E_2).$$

In other words, Z is a group-homomorphism $K_0(\mathrm{Coh}(X)) \to \mathbb{C}$. This already tells you something about the abelian category: it means that no nonzero object ends up as zero in K_0. In surfaces, there are nonzero objects that go to zero in the K-group, so you can't define the stability condition on the abelian category. However, you can get around these problems by using derived categories.

4. This should more rightfully be called a theorem. This is called the **Harder-Narasimhan property.** Before we had this definition of slope stability.

1.4 Definition. An object E is called **Z-stable** if for all nonzero proper subobjects $E' \subsetneq E$,

$$\arg Z(E') < \arg(Z(E)).$$

If $\leq$ replaces $<$, then we get the definition of **Z-semistable.**

The Harder-Narasimhan property is like a Jordan-Hölder filtration, which is a filtration of an object in an abelian category where the successive quotients are simple.

The Harder-Narasimhan property is as follows. For all E, there exists a sequence of objects

$$0 = E_0 \subset E_1 \subset \cdots \subset E_n = E,$$

such that:

- E_{i+1}/E_i is Z-semistable.
- $\arg(Z(E_{i+2}/E_{i+1})) < \arg(Z(E_{i+1}/E_i))$.

An audience member asks whether this can be proved graphically if Z takes values in a lattice. If there is a bound to the left of the degree of any subobject, then you can get this formally.

It's a theorem of Harder and Narasimhan that this holds for vector bundles. (For sheaves??)

1.5 Definition. Fix an abelian category $\mathcal{C}$. A **Bridgeland stability condition** on $\mathcal{C}$ is a map

$$Z : \mathrm{ob}\mathcal{C} \to \mathbb{C},$$

satisfying the above properties.

Now let me give you a feel for why this might be useful. Given a stability condition Z, we define

$$\phi(E) = \frac{1}{\pi} \arg(Z(E)), \quad E \in \mathcal{C},$$

and this implies that $\phi(E) \in (0, 1]$. This is called the **phase** of E.

1.6 Definition. For all $\phi \in (0, 1]$, we define a subcategory $P(\phi) \subset \mathcal{C}$ be the full subcategory spanned by (i.e., consisting of) semistable objects of phase ϕ. I will also add 0 to this subcategory. This category is closed under direct sums.

Remark. Question from audience: is a general stability condition going to respect the tensor structure in a good way, the way degree is additive for the tensor product of vector bundles and rank is multiplicative? I don't know. Almost all the examples I think about are sheaves on something which have natural tensor structures, but the tensor structure doesn't really intervene.

1.7 Proposition. *Let $E \in P(\phi)$ and $E' \in P(\phi')$. If $\phi' < \phi$, then*

$$\mathrm{Hom}(E, E') = 0.$$

In other words, maps always increase phases. You might be used to the idea of a t-structure on a triangulated category, when there's a top category and a bottom category. This is some refinement of that. Even within an *abelian* category, we can divide objects into some sort of order.

Proof. Let me give you 120 seconds to think about it...

Here's a proof. Assume there's a nonzero morphism $f : E \to E'$. Consider the image of f and its phase $\phi(\mathrm{im} f)$. We're going to form two natural short exact sequences

$$0 \to \ker f \to E \to \mathrm{Im}(f) \to 0.$$

Since E is semistable, you know that $\phi(\ker f) \leq \phi(E)$. So $\phi(\mathrm{Im} f) \geq \phi(E)$ by additivity of Z.

On the other hand, $\mathrm{Im} f \subset E'$, so by E'-semistability, it follows that $\phi(\mathrm{Im}(F)) \leq \phi(E')$. That's a contradiction by our assumptions. ▲

This is the **see-saw property.** Given any extension

$$0 \to A \to E \to B \to 0,$$

then you always end up with $\phi(A), \phi(B)$ on opposite sides of $\phi(E)$. If E is semistable, then we know $\phi(A) \leq \phi(E)$.

§5 Bridgeland stability for triangulated categories

If you don't know what a triangulated category is, you'll learn when we talk about stable ∞-categories in weeks 5-6.

1.8 Definition. I'm going to give you a somewhat dissatisfying definition. Let $\mathcal{C}$ be a triangulated category. A **Bridgeland stability condition** σ on $\mathcal{C}$ is a pair $\sigma = (Z, \heartsuit)$, where $\heartsuit$ is the heart of a bounded t-structure and Z is a Bridgeland stability condition on $\heartsuit$.

So, if you believe what I just told you, we constructed a Bridgeland stability condition on the derived category of coherent sheaves on a curve, using the usual t-structure.

Remark. There might exist $\heartsuit$'s for which there is no possible stability condition Z.

The space of stability conditions is supposed to have a topology. When I first heard a talk about this in Jerusalem, someone says "it might be clear how to topologize this in the Z-direction." However, he didn't say how to vary the heart. Someone asked "What does it mean for two hearts to be close?"

I want to construct the structure that you get starting with this.

First observation: there is a notion of a Grothendieck group of a triangulated category. Namely, you take exact triangles and declare them to be additive.

Remark.

$$K_0(\mathcal{C}) = K_0(\heartsuit),$$

if $\heartsuit$ is the heart of a t-structure on $\mathcal{C}$.

So, given a Bridgeland stability condition $(Z, \heartsuit)$, then I can define E on any object in $\mathcal{C}$, not just in $\heartsuit$. What does this thing look like? On $D^b\mathrm{Coh}(\mathbb{P}^1)$, with the usual t-structure, then $\heartsuit = \mathrm{Coh}(\mathbb{P}^1)$, and then we have that all the stable objects consists of all line bundles $\mathcal{O}(n), n \in \mathbb{Z}$, and the structure sheaves of points in $\mathbb{P}^1$. (**Did I copy this down right?**) But you can also consider shifts. Note that

$$E[1] = -E \in K_0(\mathcal{C}), \quad X \in \mathcal{C},$$

thanks to the exact triangle

$$E \to 0 \to E[1] \to .$$

You might ask yourself: I have this plot inside the complex plane. What if I rotate? The $\heartsuit$ is something that lay in the upper half-plane. We can rotate the stability condition, i.e., rotate Z. It turns out that there is a group that actions on the space of stability conditions, and you can rotate Z and get a new $\heartsuit$. Namely, when you rotate Z to Z', the objects mapped under Z' to the upper half plane and are stable form a new heart. (**Wait but what about double shifts?** There's some sort of cutoff...)

Kontsevich talked about this during the Gelfand conference. This looks like the stability condition of another category. Consider a quiver. Consider the Kronecker or Quiver quiver Q, which has two arrows from a to b. Consider the abelian category of *representations* $\mathrm{Rep}(Q)$. We define a function, called the *central charge,*

$$Z : K_0(\mathrm{Rep}(Q)) \to \mathbb{C},$$

where it turns out that $K_0(\mathrm{Rep}(Q)) \simeq \mathbb{Z}^2$. Given a representation of Q, we keep track of the dimension at each vertex.

Now take

$$Z_0 = Z(\cdot \rightrightarrows 0), \quad \cdot = k,$$

and take

$$Z_1 = Z(0 \rightrightarrows \cdot), \quad \cdot = k.$$

Let's plot them. Either the slope of Z_0 is greater than that of Z_1, or vice versa. Either of these define stability conditions on $\mathrm{Rep}(Q)$.

1.9 Exercise. Find all Z-semistable objects of $\mathrm{Rep}(Q)$ when $\arg Z_1 > \arg Z_0$.

You can plot what the stable objects look like with a given central charge. What are the stable objects of $\mathrm{Rep}(Q)$ when $\arg Z_0 > \arg Z_1$? It turns out that there are three kinds:

1. The object $0 \rightrightarrows k$ defining the number Z_1.
2. $k \rightrightarrows 0$ defining Z_0.
3. $k \rightrightarrows k$ where of the two elements $a, b \in k$, one must be nonzero for stability. Two objects are equivalent when a, b are related by a ratio: so these stable objects are parametrized by $\mathbb{P}^1(k)$.

It is a theorem of Beilinson that $D^b Coh(\mathbb{P}^1) \simeq D^b(\mathrm{Rep}(Q))$, which we haven't proved.

Lecture 2
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§1 Announcements

Email me saying whether you'd like to be on Team Edward, Team Jacob, or Team Apple. Also, Krishanu and Charmaine will be note-takers for next week.

§2 Review and corrections

I defined what a Bridgeland stability condition is for an abelian category.

2.1 Definition. A **Bridgeland stability condition** on an abelian category $\mathcal{C}$ is a function

$$Z : \mathrm{ob}\mathcal{C} \to \mathbb{C},$$

satisfying certain properties:

1. $\mathrm{im}(Z) \subset \mathbb{H} \setminus \mathbb{R}_{>0}$, the upper half plane minus the positive reals.
2. If $Z(E) = 0$, then $E = 0$. (This is already a highly nontrivial condition.)

3. If $0 \to E_1 \to E \to E_2 \to 0$ is a SES, then
$$Z(E) = Z(E_1) + Z(E_2).$$
So, Z defines a homomorphism $K_0(\mathcal{C}) \to \mathbb{C}$.[1]

4. Harder-Narasimhan property. Every object $E \in \mathcal{C}$ admits a sequence
$$0 = E_0 \subsetneq E_1 \subsetneq E_2 \subsetneq \cdots \subsetneq E_n = E,$$
such that, for each i:
   - E_{i+1}/E_i is semistable.[2]
   - $\arg(E_{i+1}/E_i) > \arg(E_{i+2}/E_{i+1})$.

   This **Harder-Narasimhan filtration** gives a sequence of objects whose Z's give vectors $Z(E_{i+1}/E_i)$ in $\mathbb{C}$ are steadily moving clockwise: the phases are decreasing. The sum of all these vectors is $Z(E)$.

Correction from last time. I gave the example of a stability condition where $\mathcal{C} = \mathrm{Coh}(\mathbb{P}^1)$ and
$$Z(E) = i\mathrm{rk}E - \deg E.$$
You can easily check that the first two conditions are satisfied. Last time I drew what the stable objects for the central charge looked like.

1. The structure sheaves at any point was stable. (These live at $-1 \in \mathbb{C}$.)

2. The line bundles $\mathcal{O}(n), n \in \mathbb{Z}$. (These live at $i + n$.)

What I claimed was that if you rotated, you'd see the category of representations of a quiver. That's not true. It is a rotation but with a small deformation. Replace Z by
$$Z(E) = (i - \epsilon)\mathrm{rk}(E) - \deg E, \quad \epsilon > 0.$$
Now, rotate Z by ninety degrees. There's one complaint here. Once I rotate, the image of Z won't live in the upper half plane. Here I'm secretly thinking of derived categories. Once you have Z defined on $K_0(\mathcal{C})$, it gets defined on the bounded derived category. Once you define Z on $\mathcal{O}$, you're also defining $\mathcal{O}[1]$. (Some pictures that I can't liveTEX.) Note that once you rotate by ninety degrees, $\mathcal{O}(-k)[1], k > 0$ have Z in $\mathbb{H}$.

2.2 Exercise. Consider the category of coherent sheaves on $\mathbb{P}^1$.

- Compute $\mathrm{Ext}^\bullet(\mathcal{O} \oplus \mathcal{O}(1), \mathcal{O} \oplus \mathcal{O}(1))$. This is concentrated in degree zero and is some algebra A.

[1] For sheaves on complex surfaces, the obvious stability function can send nontrivial objects to zero, so not every abelian category can satisfy this.

[2] Recall this definition from last time.

- Show that the category of A-modules is equivalent to the category of representations of the Kronecker quiver $\cdot \rightrightarrows \cdot$ (this is as abelian categories). To see this, construct the path algebra, which has an idempotent for every vertex, and a generator for every edge. You'll find that this algebra is precisely A.

- Exhibit a functor
$$D^b\mathrm{Coh}(\mathbb{P}^1) \to D^b\mathrm{Rep}(\cdot \rightrightarrows \cdot).$$
Namely, take
$$E \to \mathrm{Ext}^\bullet(\mathcal{O} \oplus \mathcal{O}(1), E).$$

You should do this at home. You don't need to prove it is an equivalence. This is a theorem of Beilinson. The point is that $\mathcal{O}, \mathcal{O}(1)$ are enough to generate the derived category of coherent sheaves. This should be one of your favorite examples of Bridgeland stability conditions.

Last time, I defined a stability condition on the (ordinary) category of quiver representations, which I claimed recovered this rotated stability condition on coherent sheaves of $\mathbb{P}^1$. I claimed, consider

$$Z : \mathrm{Rep}(Q) \to \mathbb{C},$$

which is determined by where it sends $k \rightrightarrows 0$ (say, to Z_0) and where it sends $0 \rightrightarrows k$ (say, to Z_1). If $\arg Z_0 > \arg Z_1$, then you realize that there's a moduli space of stable objects which is $\mathbb{P}^1$ (the skyscraper sheaves)? Answer: $\mathcal{O}$ goes to $k \rightrightarrows 0$ and $\mathcal{O}(-1)[1]$ gets sent to $0 \rightrightarrows k$. Somehow, it reverses the orientation.

Remark. We will see that there is an action of the universal cover $\widetilde{GL_2(\mathbb{R})}$ on the space of stablity conditions on a triangulated category.

2.3 Definition. Let $\mathcal{C}$ be a triangulated category. A **Bridgeland stability condition** on $\mathcal{C}$ is a pair $\sigma = (Z, \heartsuit)$ where:

1. $\heartsuit$ is the heart of a bounded t-structure on $\mathcal{C}$ (which determines the t-structure).

2. Z is a Bridgeland stability condition on $\heartsuit$.

Last time, I defined categories $\mathcal{P}(\phi)$ for $\phi \in (0, 1]$ such that

$$\mathcal{P}(\phi) \subset \heartsuit,$$

consisting of Z-semistable objects in $\heartsuit$ with *phase* ϕ (i.e., $\frac{1}{\pi} \arg Z(\cdot)$ is the phase). This actually defines a full subcategory of $\mathcal{C}$ for all real numbers $\phi \in \mathbb{R}$. Namely,

$$\mathcal{P}(\phi + n) \stackrel{\text{def}}{=} \mathcal{P}(\phi)[n].$$

So we get a family of subcategories indexed by $\mathbb{R}$.

Because of the Harder-Narasimhan property, this gives a decomposition of every object in $\mathcal{C}$. Given any $X \in \mathcal{C}$, write X as an extension of objects that live within one interval, and then do the Harder-Narasimhan within each interval. This leads to:

2.4 Definition. A **slicing** for a triangulated category $\mathcal{C}$ is the following data: a full additive subcategory $\mathcal{P}(\phi) \subset \mathcal{C}$ for each $\phi \in \mathbb{R}$.[3] We require the following conditions:

1. If $\phi > \phi'$, then if $E \in \mathcal{P}(\phi), E' \in \mathcal{P}(\phi')$, then
$$\mathrm{Hom}_{\mathcal{C}}(E, E') = 0.$$
2. We require an analog of the Harder-Narasimhan property. Given $E \in \mathcal{C}$, we require a filtration
$$0 = E_0 \to E_1 \to \cdots \to E_n \to E,$$
such that $E_{i+1}/E_i \in \mathcal{P}(\phi_{i+1})$, where
$$\phi_0 > \phi_1 > \cdots > \phi_n.$$
3. $\mathcal{P}(\phi + 1) = \mathcal{P}(\phi)[1]$.

2.5 Definition. Here is another definition of a stability condition. A **Bridgeland stability condition** on $\mathcal{C}$ (a triangulated category) is a pair $(Z, \mathcal{P})$ where
$$Z : K_0(\mathcal{C}) \to \mathbb{C}$$
is a homomorphism, $\mathcal{P}$ is a slicing of $\mathcal{C}$, and where they satisfy the following compatibility condition. For all $E \in \mathcal{P}(\phi)$,
$$Z(E) = me^{i\pi\phi}, \quad m \in \mathbb{R}_{>0}.$$

2.6 Proposition. *The two definitions of a Bridgeland stability condition are equivalent.*

Proof. Sketch. Given a Bridgeland stability condition in the second sense, we define $\heartsuit$ to be the smallest collection of objects in $\mathcal{C}$ consisting of $\{\mathcal{P}(\phi)\}_{\phi \in (0,1]}$ and closed under extensions. ▲

2.7 Theorem. *Under some conditions, consider the collection* $\mathrm{Stab}(\mathcal{C})$ *of stability conditions on* $\mathcal{C}$*. Then the forgetful map*
$$\mathrm{Stab}(\mathcal{C}) \to \mathrm{Hom}(K_0(\mathcal{C}), \mathbb{C})$$
is a local homeomorphism.[4]

[3] Remark: semistable objects are closed under direct sum.
[4] For an appropriate definition of the topology on $\mathrm{Stab}(\mathcal{C})$.

§3

Let's talk about homological mirror symmetry. In physics, consider $N = (2, 2)$ SUSY 2-D CFT. So this means you have a manifold X, which is a Calabi-Yau threefold. At the very foundations of supersymmetry (which is some sort of $\mathbb{Z}/2$-graded Lie algebra), if you take as a given that whatever laws of physics should be invariant under certain symmetries, then you can write down equations that should be invariant under them. The symmetry implies that X has some sort of structure. That's how "Calabi-Yau" pops up. If you like the language of TFT's, you know that's some theory that gives an invariant of manifolds up to diffeomorphism or homeomorphism. A CFT is also sensitive to a conformal structure. As a mathematician, you should imagine that once you get X, you should get an invariant of *Riemann* surfaces, rather than genus g surfaces.

Is there a way of getting an invariant which is only sensitive to the topological structure? There's a topological twist that you can do to the conformal field theory. It's some sort of physics term. Where does "twist" come from? Everything comes from some sort of $U(1) \times U(1)$ symmetry. There are two kinds of twists and two kinds of field theories. There are called the A-model and the B-model. So you observe that if you have a Calabi-Yau manifold, then you have a $\mathbb{C}$-structure and a symplectic structure, which a lot of physicists think of as the Kähler metric. The field theory that you get from the A-model doesn't depend on the complex structure. But for the B-model, it doesn't depend on the symplectic structure. Physicists tried to give a mathematical meaning to the B-model. "The B-model is the derived category of coherent sheaves on X." What does a physicist think about that? What does that mean? "Somehow you write out the things it should satisfy and it's the derived category."

If you want to make sense of what it means for strings to propagate on X, then the B-model corresponds to $D^b\text{Coh}(X)$, while the A-model is supposed to be the Fukaya category. For reasons that I don't understand, there's supposed to be a target called $X^\vee$ that has the same theory. Here's a mathematically precise statement, called the **homological mirror symmetry conjecture**:

> **Conjecture (rough):** Let's say that X is a complex Kähler manifold. There exists a Kähler $X^\vee$ such that:
>
> 1. $\text{Fuk}(X) \simeq D^b\text{Coh}(X)$.
> 2. $D^b\text{Coh}(X) \simeq \text{Fuk}(X^\vee)$.
>
> I haven't defined what the Fukaya category is. It's some exchange of complex and symplectic structure between X and $X^\vee$.

If we fix X, there is a moduli space of complex structures on X, and of symplectic structures on X. This latter (with the complex structure fixed) might be called a Kähler moduli space by physicists. Same for $X^\vee$. For X, consider the Kähler moduli space Kahler(X). As I move around in Kahler(X), I get a family of B-models (derived categories) that don't change. In $X^\vee$, then I'm moving around in the moduli space of complex structures. (**Completely lost here...**)