BRIDGELAND STABILITY
HIRO LEE TANAKA
LECTURES 1 AND 2

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Notes taken by Lukas Brantner

CONTENTS

Organisational Preliminaries. There will be three groups of talks:
- Team Edward - weeks 4-5: $Coh(X)$ as an abelian category
- Team Jacob - weeks 5-6: (stable) ∞-categories
- Team Apple - final two weeks: Advanced topics such as applications to minimal model program, the reconstruction problem, applications to Mirror Symmetry and dynamical systems

1. COHERENT SHEAVES ON CURVES

Fix a smooth projective curve X over the field $\mathbb{C}$ of complex numbers and assume that we are given a vector bundle E on X. We can associate two different invariants:
- The rank $rk(E)$ of E, i.e. the dimension of each fibre.
- The degree $deg(E)$ of E, which is the number "zeros - poles" of a generic meromorphic section of its top exterior power $\Lambda^{rk(E)}E$

We can now combine these two invariants into one:

Definition 1. The *slope* of a vector bundle is defined to be
$$\mu(E) := \frac{deg(E)}{rk(E)}$$

We introduce the notion of stability:

Definition 2. A vector bundle E on X is said to be *stable* (*semistable*) if for all proper nonzero subbundles $F \subset E$, we have
$$\mu(F) < \mu(E)$$
$$(\mu(F) \leq \mu(E))$$

The upshot is that once we restrict attention to semistable vector bundles on X, we get a nice modular description. More precisely, Mumford showed that there is a coarse moduli space which is a quasiprojective variety.

Remark 3. One can easily extend the notions of rank and the degree to the context of quasicoherent sheaves on (irreducible, reduced projective) *curves* [1]:
Simply define
$$rk(\mathcal{F}) := dim_{k_p}(\mathcal{F}_p/\mathfrak{m}_p\mathcal{F}_p)$$
to be the dimension of the fibre of a generic point and set
$$deg(\mathcal{F}) = \chi(\mathcal{F}) - \chi(\mathcal{O}_X) \cdot rk(\mathcal{F})$$
Here χ stands for the Euler characteristic $\chi(\mathcal{F}) = \sum_i \dim H^k(X, \mathcal{F})$.

In the last paragraph, we have first defined a notion of "stability" and then hinted at a result on the moduli of all semistable objects (of a given rank and degree).
In the following, we will try to generalise these two steps to more general categories. First, we will introduce Bridgeland's notion of "stability condition" on an abelian or triangulated category. The moduli space of all such conditions will form a complex manifold.
Once we fix such a stability condition, we obtain another moduli space of semistable objects. One aim of this theory is to understand the geometry of this space and how it varies with the chosen stability condition.

Remark 4. By passing to the quotient of rank and degree, we have lost information. We shall therefore keep track of the pair $(rank, degree)$ rather than just their ratio. We will try to find the important abstract properties of this pair which we can generalise in a second step.

This leads to the following observation:

Proposition 5. *Let $\mathcal{C}$ be the category of coherent sheaves on the smooth projective curve X. Consider the function*
$$Z : ob\,\mathcal{C} \to \mathbb{C}$$
$$E \mapsto -deg(E) + i \cdot rk(E)$$
This function has the following properties:
- *0) $Z(E) \subset \{z \in \mathbb{C} | Im(z) > 0 \text{ or } Im(z) = 0 \text{ and } Re(z) \leq 0\}$*
  (This uses that Torsion sheaves always have nonnegative degree)
- *1) $Z(E) = 0$ if and only of $E = 0$*
- *2) Z is additive on short exact sequences in the abelian category $Coh(X)$.*
  (This means that we have a group homomorphism $K_0(\mathcal{C}) \to (\mathbb{C}, +)$.)
- *3) Z has the Harder-Narasimhan property.*

The Harder-Narasimhan property is a mirror of the existence of Harder-Narasimhan filtrations for coherent sheaves on curves, which is unsurprisingly a theorem of

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[1] The definition of degree on higher-dimensional varieties is more involved and requires the choice of a Kähler form.

Harder and Narasimhan:
Given any coherent sheaf E, there is a unique sequence of subobjects

$$0 = E_0 \subset E_1 \subset ... \subset E_n = E$$

such that the successive quotients E_{i+1}/E_i are semistable and of strictly decreasing slope.

2. Bridgeland Stability Conditions on Abelian Categories

Using that $\mu(F) < \mu(E)$ if and only of $arg(Z(F)) < arg(Z(E))$, we can easily express this purely in terms of the function Z:

Definition 6. Given an abelian category $\mathcal{C}$ with a function $Z : \mathcal{C} \to \mathbb{C}$. An object $E \in ob\,\mathcal{C}$ is $Z-$stable (Z-semistable) if for all proper nonzero subobjects $F \subset E$, we have

$$Arg(Z(F)) < Arg(Z(E))$$
$$(Arg(Z(F)) \leq Arg(Z(E)))$$

Definition 7. We say that the function Z has the Harder-Narasimhan property if for any object E, there is a sequence of subobjects

$$0 = E_0 \subset E_1 \subset ... \subset E_n = E$$

such that the successive quotients are $Z-$stable and have decreasing $Z-$argument.

We can now finally introduce Tom Bridgeland's notion of "stability condition":

Definition 8. Let $\mathcal{C}$ be an abelian category. A *Bridgeland stability condition* on $\mathcal{C}$ is an additive map

$$Z : K_0(\mathcal{C}) \to \{z \in \mathbb{C} | Im(z) > 0 \text{ or } Im(z) = 0 \text{ and } Re(z) \leq 0\} \subset \mathbb{C}$$

which sends exactly the zero bundle to 0 and satisfies the Harder-Narasimhan property.

Exercise 9. *The Harder-Narasimhan filtrations are unique.*

We have already seen that the slope is mirrored by the argument of Z, and we therefore give it a special name:

Definition 10. Given a pair $(\mathcal{C}, Z)$ of abelian category and complex-valued function, we write

$$\phi(E) = \frac{1}{\pi} Arg(Z(E)) \in (0, 1]$$

and call it *the phase.* Note that this is only well-defined on objects with $Z(E) \neq 0$.

The contours of the potential divide up our category:

Definition 11. Let $(\mathcal{C}, Z)$ be a Bridgeland stability condition. Given any $\phi \in (0, 1]$, we define

$$P(\phi) \subset C$$

to be the full subcategory (not necessarily closed under extensions!) spanned by the zero object and all $Z-$semistable objects of phase ϕ.

Remark 12. These categories are closed under direct sums.

Whenever we have a $t-$structure in a triangulated category, we have divided it up into two parts $(\mathcal{C}^{\leq 0}, \mathcal{C}^{\geq 1})$ and impose the condition that there are no nontrivial homomorphisms from objects in the nonpositive to objects in the positive half.

We observe that the contours of a Bridgeland stability condition satisfy an analogous property:

Proposition 13. *Let $E \in P(\phi)$ and $E' \in P(\phi')$ be two semistable objects whose phases satisfy*

$$\phi > \phi'$$

Then $Hom_{\mathcal{C}}(E, E') = 0$.

Proof. This is a good illustration of the very handy "seesaw-property", which is often a helpful tool in these kinds of proofs.

The basic observation is that given any short exact sequence of nonzero objects

$$0 \to A \to E \to B \to 0$$

the additivity equation $Z(E) = Z(A) + Z(B)$ implies that the number $\phi(E)$ must lie "between" the numbers $\phi(A)$ and $\phi(B)$ (in either order). If moreover the middle term is semistable, then we also have $\phi(A) \leq \phi(E)$ and hence $\phi(E) \leq \phi(B)$.

We now use that to prove our theorem as follows: Given a nonzero morphism

$$f : E \to E'$$

we have a short exact sequence

$$0 \to ker(f) \to E \to im(f) \to 0$$

Either $ker(f) = 0$, in which case $E \subset E'$ and we get a contradiction by semistability of E', or the Sesaw-property implies that $\phi(E) \leq \phi(im(f))$. By semistability of E', we have $\phi(im(f)) \leq E'$, we therefore conclude that

$$\phi(E) \leq \phi(E')$$

$\square$

3. Bridgeland Stability Conditions on Triangulated Categories

We saw above that on curves, we can define a natural Bridgeland stability condition by simply setting

$$Z(E) = -deg(E) + i \cdot rk(E)$$

Once we pass to surfaces such as $\mathbb{P}^2$ (and have extended our notion of rank), we see that this simple-minded definition fails since skyscraper sheaves have vanishing rank and degree.

One can in fact show that there are no stability conditions on the abelian category $Coh(X)$. The way in which to solve this problem is to pass to the derived category $D^b Coh(X)$, where we will indeed be able to spot such functions (for other $t-$structures):

Definition 14. Let $\mathcal{C}$ be a triangulated category. A *Bridgeland stability condition* σ on $\mathcal{C}$ is a pair

$$\sigma = (Z, \heartsuit)$$

where

- $\heartsuit$ is the heart of a bounded $t-$structure on $\mathcal{C}$
- Z is a Bridgeland stability condition on $\heartsuit$.

While it is still impossible to define a stability condition on $D^b Coh(\mathbb{P}^2)$ with its usual heart, we can choose a different $t-$structure, which then suddenly supports a $t-$structure.

Later, we will want to topologise the space of stability conditions.

Remark 15. Note that we can define the Grothendieck group of a triangulated category by declaring its exact triangles to be additive - we then obtain that $K_0(\mathcal{C}) \cong K_0(\heartsuit)$. Since a Bridgeland stability condition on C is just a function $K_0(\heartsuit) \to \mathbb{C}$ with some extra properties, we can use this observation to define stability conditions on triangulated categories to be functions $K_0(\mathcal{C}) \to \mathbb{C}$ satisfying similar axioms.

Example 16. Let us now examine the simplest example $\mathbb{P}^1$.
Set $\mathcal{C} = D^b(Coh(\mathbb{P}^1))$ and put the usual $t-$structure and stability-condition

$$Z(E) = -deg(E) + i \cdot rk(E)$$

on it.
One can prove that a sheaf in the heart $\heartsuit = Coh(\mathbb{P}^1)$ is stable if and only if it is either a Serre-twist of the structure sheaf or a skyscraper sheaf:

$$\{\mathcal{O}(n)\}_{n \in \mathbb{Z}}$$

$$\{(\iota_x)_*(k)\}_{x \in \mathbb{P}^1}$$

The exact triangle

$$E \to 0 \to E[1]$$

shows that

$$[E[1]] = -[E] \in K_0(\mathcal{C})$$

The function function Z has the following form on stable objects:
- It takes the sheaf $\mathcal{O}(n)$ to $n + i$
- The skyscraper sheaves $\mathcal{O}_x$ go to (-1) on the negative real line.

Using the additive extension of the function Z to the derived category, we see that

$$Z([E]) = -Z([E[1]])$$

which gives us more complex numbers in the image of Z in the obvious fashion.

4. The Kronecker Quiver

A quiver Q is just a directed multigraph in the context of representation theory. A $k-$linear representation of such a Quiver assigns to each vertex a $k-$vector space and to each directed edge a linear map.

We define the *path algebra* $k\Gamma$ to be the free $k-$vector space on the collection Γ of all *paths* in Q - concatenation gives rise to multiplication. Notice that representations of a Quiver and representations of its path-algebra are equivalent notions.

The upshot is that we can compare categories of representations of certain quivers to categories of coherent sheaves on certain varieties. The following theorem of Beilinson gives a first example of this:

Theorem 17. *Write $Q = (\bullet \rightrightarrows \bullet)$ for the Kronecker Quiver. Then*

$$D^b(Coh(\mathbb{P}^1)) \cong D^b(Rep(Q))$$

Exercise 18. *In this exercise, we will outline how the map from sheaves to representations in the above equivalence is constructed:*

- *Show that*
$$Ext^* \left(\mathcal{O} \oplus \mathcal{O}(1), \mathcal{O} \oplus \mathcal{O}(1)\right)$$

   is concentrated in degree 0. Write A for this algebra.
- *Show that A is exactly the path-algebra of the Kronecker quiver Q.*
- *Use this to produce a functor $D^b(Coh(\mathbb{P}^1)) \to D^b(Rep(Q))$*

The theorem of Beilinson shows that $D^b(Coh(\mathbb{P}^1))$ is generated by $\mathcal{O}$ and $\mathcal{O}(-1)$, and a similar statement holds for higher-dimensional projective spaces.

Given the general context in which Bridgeland stability conditions are defined, it is natural to wonder what they look like on the category $D^b(Rep(Q))$.

Observe that
$$K_0(Rep(Q)) \cong \mathbb{Z}^2$$
is generated by the representations
$$R_0 = (k \rightrightarrows 0)$$

$$R_1 = (0 \rightrightarrows k)$$
So once we fix a stability condition $Z : K_0(Rep(Q)) \to \mathbb{C}$, we obtain two complex numbers $Z_i = Z(R_i) \in \mathbb{C}$.

Exercise 19. *Find all $Z-$semistable objects of $Rep(Q)$ in the situation where $arg(Z_1) > arg(Z_2)$.*

When $arg(Z_0) > arg(Z_1)$, the stable objects in $Rep(Q)$ are of the following form:

- $0 \rightrightarrows k$. This corresponds to $\mathcal{O}(-1)[1]$ under Beilinson's isomorphism.
- $k \rightrightarrows 0$. This corresponds to $\mathcal{O}$ under Beilinson's isomorphism.
- $k \rightrightarrows k$, where the arrows correspond to multiplication by elements a, b of k, one of which has to be nonzero, and two such representations are equivalent iff $[a : b] = [a' : b']$. In other words, this family of stable representations is parametrised by $\mathbb{P}^1$, these correspond exactly to the skyscraper sheaves.

We see that such a stability condition on $D^b(Rep(Q))$ is not compatible with the usual stability condition on $D^b(Coh(\mathbb{P}^1))$ under Beilinson's isomorphism in the weak sense that stable objects in the heart don't match up. We can fix this by defining a new stability condition on $D^b(Coh(\mathbb{P}^1))$ as follows:

- Shear Z slightly in the horizontal direction, i.e. take $\epsilon > 0$ small and define
$$Z'(E) = -deg(E) + (i - \epsilon)rk(E)$$
- Rotate Z' by 90 degrees to obtain Z''

Once we carry out these two operations, different elements in $D^b(Coh(X))$ land in the upper half plane, e.g. $\mathcal{O}(-k)[1]$ for $k > 0$.

Remark 20. We will see later that the universal cover $\overline{GL_2(\mathbb{R})}$ acts on the space of all stability conditions of a triangulated category.

5. Slicing

Previously, we have defined the notion of a Bridgeland stability condition on a triangulated category by simply defining it on its heart. In order to rephrase this definition, we will start with a triangulated category $\mathcal{C}$, endowed with a $t-$structure and a Bridgeland stability condition

$$Z : \heartsuit \to \mathbb{C}$$

and spot a new structure we can define on C.

The function Z naturally extends to all of $\mathcal{C}$. Previously, we have used the phase

$$\phi = \frac{1}{\pi} arg \circ Z(-) \in (0, 1]$$

to divide up our heart into many different pieces, indexed by the half-open interval $(0, 1]$. We can easily extend this strategy to all of $\mathcal{C}$ by defining

$$\mathcal{P}(\phi + n) := \mathcal{P}(\phi)[n]$$

This gives a family of full additive subcategories of $\mathcal{C}$ indexed by the real numbers.

We can first decompose objects in $\mathcal{C}$ as extensions of objects living in various shifts of the heart, and then use the Harder-Narasimhan property of Z these pieces. This shows that the family

$$\{\mathcal{P}(\phi)\}_{\phi \in \mathbb{R}}$$

satisfies the following axioms:

Definition 21. A *slicing* P of a triangulated category $\mathcal{C}$ is a family

$$\{P(\phi) \subset C\}_{\phi \in \mathbb{R}}$$

of full additive subcategories satisfying:

- 0) $P(\phi + 1) = P(\phi)[1]$
- 1) If $\phi > \phi'$, then for any two objects $E \in P(\phi), E' \in P(\phi')$, we have
$$Hom_{\mathcal{C}}(E, E') = 0$$

- 2) We have the Harder-Narasimhan property, by which we mean that for all $E \in ob\ \mathcal{C}$, there is a sequence
$$0 = E_0 \subset E_1 \subset ... \subset E_n = E$$
such that
$$E_{i+1}/E_i \in P(\phi_{i+1})$$
and
$$\phi_0 > \phi_1 > ... > \phi_n$$

With this notation, we can now give a more pleasing definition of Bridgeland stability on triangulated categories:

Definition 22. A *Bridgeland stability condition* on a triangulated category is a pair

$$(Z, P)$$

where

- $Z : K_0(\mathcal{C}) \to \mathbb{C}$ is a homomorphism
- P is a slicing

These two objects are compatible in the sense that for all $E \in P(\phi)$, we have

$$Z(E) = m \cdot e^{i\pi\phi}$$

for some $m \in \mathbb{R}_{\geq 0}$.

Proposition 23. *Both definitions of Bridgeland stability conditions are equivalent.*

Proof. (Sketch) In order to recover the heart from a Bridgeland stability condition in the second sense, we simply let $\heartsuit$ be the smallest full subcategory containing all the objects in $\{P(\phi)\}_{\phi \in (0,1]}$ and being closed under extensions. $\square$

We will now state a vague version of an important theorem on the moduli of stability conditions (whose topology we still have to define):

Theorem 24. *Under some conditions, the map*

$$Stab(\mathcal{C}) \to Hom(K_0(\mathcal{C}), \mathbb{C})$$

$$(Z, P) \mapsto Z$$

is a local homeomorphism.

6. A Fairy Tale about the Homological Mirror Symmetry Conjecture

In physics, we can consider $(N = 2, 2)$ SUSY two-dimensional conformal field theories (these are meant to give invariants for Riemann-surfaces, i.e. take the conformal structure into account). Each of these theories has a *target manifold* X attached to it, and this is in fact a Calabi-Yau threefold (this is related to supersymmetry).

We can carry out two "topological twists" to such a theory (these come from symmetries of a $U(1) \times U(1)-$action), and these twists give rise to two field theories: The A-model and the B-model.

- The A-model does not depend on the complex structure J of the Calabi-Yau threefold - it will "correspond" to $D^b(Coh(X))$.
- The B-model does not depend on the Kähler form ω of the Calabi-Yau threefold and will "correspond" to the Fukaya-category $Fuk(X)$.

The homological mirror symmetry conjecture says:

Conjecture 25. *(Originally due to Kontsevich) Under many conditions, given a Kähler manifolds X, there is another Kähler manifold $X^\vee$ such that*

$$Fuk(X) \cong D^b(Coh(X^\vee))$$

$$D^b Coh(X) \cong Fuk(X^\vee))$$

If we fix X, there is a moduli space of complex and a moduli space of symplectic structures on X, and similarly for its mirror $X^\vee$. One can now consider paths in these moduli spaces and examine how they act on the various attached categories attached above.

7. A theorem of T.Toda related to the Minimal Model Program

Let us quote an important theorem of Tukinobu Toda illustrating the usefulness of Bridgeland stability conditions.

Let X be a smooth complex projective *surface.* Restricting attention to the space of stability conditions $Stab_N(X)$ which factor through the *numerical Grothendieck group* [2] , we obtain a local homeomorphism

$$\Pi : Stab^N(X) \to N(X)_{\mathbb{C}}^{\vee}$$

We can pull this map back against

$$NS(X)_{\mathbb{R}} \xrightarrow{-\int_X e^{-i*}} N(X)_{\mathbb{C}}^{\vee}$$

to obtain a space

$$Stab(X)_{\mathbb{R}}^N$$

Given a stability condition $\sigma = (Z, \heartsuit)$, write $\mathcal{M}^{\sigma}([\mathcal{O}_x])$ for the space of Z−stable objects E in the heart with phase $\phi = 1$ and $Z(E) = Z(\mathcal{O}_x)$ for all $x \in X$.

We then have the following result:

Theorem 26. *(Toda) For all smooth projective surfaces Y and all birational morphisms*

$$f : X \to Y$$

there is

$$U(Y) \subset Stab_{\mathbb{R}}$$

open such that

- *For all $\sigma \in U(Y)$, we have $\mathcal{M}^{\sigma}([\mathcal{O}_x]) \cong Y$*
- *If f factors through a blowup at a point $Y' \to Y$, then*

$$\overline{U}(Y) \cap \overline{U}(Y') \neq \emptyset$$

From this, we can conclude the following result:

Corollary 27. *(Toda) Let X be a smooth complex projective surface and assume*

$$X = X_1 \to X_2 \to ... \to X_N$$

is a minimal model program[3] for X.
Then there is a continuous family

$$\{Z_t\}_{t \in (0,1)}$$

of Bridgeland stability conditions and real numbers

$$0 = t_0 < t_1 < ... < t_N = 1$$

such that we have an isomorphism

$$X_i \cong \mathcal{M}^{Z_t}([\mathcal{O}_x])$$

for all $t \in (t_{i-1}, t_i)$.

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[2]The numerical Grothendieck group can be obtained from the usual Grothendieck group $K_0(X)$ by dividing out all $\mathcal{E}$ for which the Euler form

$$\chi(\mathcal{E}, \mathcal{F}) = \sum_i (-1)^i dim Ext^i(\mathcal{E}, \mathcal{F})$$

vanishes for all $\mathcal{F}$

[3]In the case of a surface, all the maps involved are contractions of (-1)-curves.