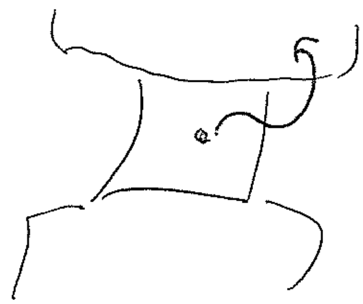
Some physical history.

'93 Aspinwall-Greene-Morrison

(old) Kähler moduli space of X — Complex structures moduli space of $X^\vee$

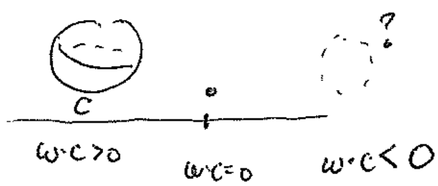
Guided by mirror symmetry, we move around in $\mathbb{C}$-moduli space of $X^\vee$. What's resulting effect in X theory?

"new", expanded Kähler space

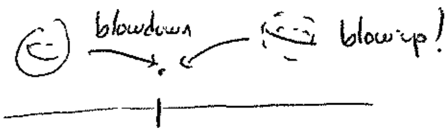


Depends on the kinds of <u>walls</u> we cross, but it turns out

- Kähler class ω can turn the volume of a curve from positive to negative



Interpretation:



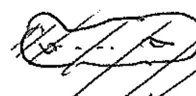
- Get new orbifolds
- etc.

Algebraic Varieties by Kodaira dimension

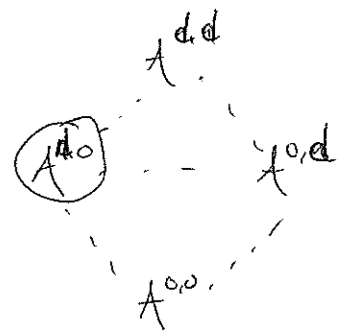
Let X be a projective variety. ($\mathbb{C}$ submfld of some $\mathbb{P}^n$, for simplicity, if you're a topologist.) Then you can associate to X a number,

$$\kappa(X) \in \{-\infty\} \cup \{0,1,2,3,\cdots n\}$$

where $n = \dim_{\mathbb{C}} X$. A table:

dim	$\kappa=-\infty$	$\kappa=0$	$\dim X > \kappa \geq 1$	$\kappa = \dim X$
1	 $\kappa=-\infty$	 $\kappa=0$	 ~~$g \geq 2$ $\kappa=1$.~~	genus ≥ 2 $\kappa=1$. 
whatever	• X is birational to $\mathbb{P}^n$ $\left(\begin{array}{c} \text{open } U \subset X \\ \cong \\ V \subset \mathbb{P}^n \\ \text{open} \end{array}\right)$ • del Pezzo surfaces $\cup$ • Fano varieties (ie, K_X^{-1} ample)	Abelian varieties + Calabi-Yau varietes	. . ? . . .	"general type" by definition Ex: $X \subset \mathbb{P}^N$ hypersurface is general type whenever $\deg > N+1$. Ex K_X ample.

Rmk This won't matter, but for those interested, let $K_X = \Omega_X^{\dim X}$ be line bundle of holomorphic top-forms on X. Then $H^0(X, K_X^{\otimes n})$ defines a sequence of vector spaces, $n=0,1,\cdots$. Just by multiplying ~~sections and~~ transition functions, we get maps $H^0(X, K_X^{\otimes n}) \otimes H^0(X, K_X^{\otimes m}) \to H^0(X, K_X^{m \otimes n})$ defining a graded ring $\bigoplus_{n \geq 0} H^0(X, K_X^{\otimes n})$.



Proj of this graded ring is a variety of some dimension, κ.

↑ Defn of Kodaira dimension.

Ex. If X is Calabi-Yau, so $K_X \cong \mathcal{O}_X$, show $K = 0$.

Goal of minimal model program (for general type X)

Find X' s.t.

- X is birational to X'
- $\forall$ curves $C \subset X'$, $\int_C K_X$, or $K_X \cdot C$, is ≥ 0.

(Curves: Trivial
Surfaces: Castelnuovo's thm; just contract $C \cong \mathbb{P}^1$ w/ negative $K \cdot C$.)

An awesome result of Toda '12

Roughly, Toda examines "the real part" of the complex mfld $\operatorname{Stab} \mathcal{C}$, when $\mathcal{C} = D^b \operatorname{Coh} X$ for X an algebraic surface.

$$\begin{tikzcd} \operatorname{Stab}_{\mathbb{R}} \arrow[r, hook, "\text{Tp.b.}"'] \arrow[d] & \operatorname{Stab}(X) \arrow[d] & (Z, \sigma) \arrow[d, mapsto] \\ NS(X) \otimes \mathbb{R} \arrow[r] & \hom(N(X), \mathbb{C}) & Z \end{tikzcd}$$

the real part

$\operatorname{Pic}(X)/\operatorname{Pic}^0(X)$ ← not sure how to capture purely categorically; Pic may use $\otimes$.

numerical Grothendieck grp. $K(X)/\operatorname{Null}(\text{Mukai})$, purely categorical

$\forall \ \sigma \in \operatorname{Stab}(X)$, let $\mathcal{M}^\sigma([\mathcal{O}_x])$ denote moduli space of $E \in \operatorname{ob} \mathcal{C}$ s.t.

- E is σ-stable
- Chern character of E = Chern character of $\mathcal{O}_x$.

Thm (Toda '12) Let X be smooth, proj. surface

$\forall\ Y$ smooth, projective, and $\forall$ birational equivalences

$$f : X \to Y$$

$\exists \quad \mathcal{U}(Y) \subset \mathrm{Stab}_{\mathbb{R}}$ s.t.

(1) If $\exists$

$$\begin{tikzcd} Y & X \arrow[l, "f"'] \\ \tilde{Y} \arrow[u] \arrow[ur, dashed] & \end{tikzcd}$$

where $\tilde{Y}$ is a blowup at a point,

then $\overline{\mathcal{U}}(Y)$ and $\overline{\mathcal{U}}(Y')$ intersect.

(2) $\forall\ \sigma \in \mathcal{U}(Y)$,

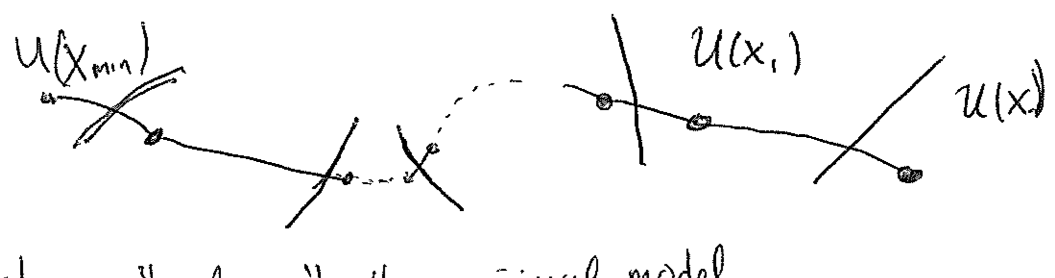
$$\mathcal{M}^{\sigma}([\mathcal{O}_x]) \cong Y.$$

(3) $\mathcal{U}$ is open and non-empty ⟵ ample cone of Y (bundle ~~open sub~~ open; open)

For surfaces, minimal model is just a sequence of removing $\mathbb{CP}^1$; so

$$\begin{tikzcd} X \arrow[d] \arrow[rrrr, dashed] & & & & \tilde{X}_1 = X_0 \\ X_{\text{minimal}} & \widetilde{X_{\text{minimal}}} = X_{n-1} \arrow[l] & \tilde{X}_{n-1} = X_{n-2} \arrow[l] & \cdots \arrow[l] & \tilde{X}_1 = X_0 \arrow[l] \end{tikzcd}$$

gives



a path in $\mathrm{Stab}_{\mathbb{R}}$ "realizing" this minimal model.

What's cool

- dim $\mathrm{Stab}(\mathcal{C})$ often seems to be <u>twice</u> the dimension of the ~~dim~~ "physical" stability condition space. (According to a mysterious remark of Kontsevich; perhaps this is well-known.) Toda has picked out a space of correct "physical" dimension

- Physicists (Aspinwall, Green, etc.) often only consider <u>Calabi-Yau</u> (albeit in 3-dim case) varieties. This is an example of a departure from C-Y cases.

So, in particular, for X a surface, the triangulated category $D^b Coh(X)$, in the abstract, can reconstruct X as a moduli space of certain kinds of objects. This depends on the correct choice of $\sigma \in Stab_{\mathbb{R}}(\mathcal{E})$, and it's not clear how to find this particular σ. (Note there's no reason to expect a "preferred" σ, since there could be many X' s.t. $D^b Coh(X') \simeq D^b Coh(X)$.)

<u>Thm</u> (Bondal-Orlov '97)

Let X be smooth, irreducible projective variety s.t. either K_X is ample, or K_X^{-1} is ample.

If $D^b Coh(X) \simeq D^b Coh(X')$, then $X \cong X'$ as a variety.

So $D^b Coh$ is a complete invariant for X in this case. We know this can't hold for all X — $\exists$ varieties related by flops for which $D^b Coh$ are equivalent.

Digressions on being ample.

<u>Defn.</u> A line bundle $\mathcal{L}$ on X is <u>ample</u> if $\forall$ coherent sheaves $\mathcal{F}$, $\exists$ n_0 s.t. $\forall$ $n \geq n_0$,

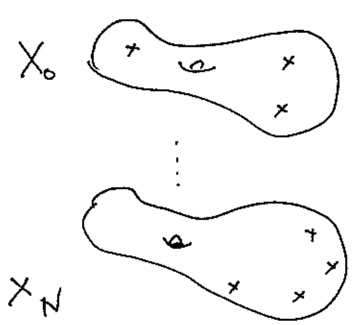
$$\mathcal{F} \otimes \mathcal{L}^{\otimes n}$$

is generated by global sections.

<u>Defn</u> A sheaf $\mathcal{F}$ is <u>generated by global sections</u> if $\exists$ finite collection $s_0, \dots, s_N \in \Gamma(\mathcal{F})$ s.t. $\forall$ $p \in X$, the stalk $\mathcal{F}_p$ is generated by $(s_0)_p$ as a module over $\mathcal{O}_p$.

What's the point? Ample line bundles yield morphisms to $\mathbb{P}^N$!

<u>Construction:</u> Given $s_0, \dots, s_N$, let $X_i = X \setminus \{\text{zeroes of } s_i\}$.



Since s_i generate stalks of $\mathcal{F}$,

$X = \bigcup X_i$. ($\nexists$ $p \in X$ s.t. $s_i|_p = 0$ $\forall i$).

Let $\mathbb{A}^N \cong U_i \subset \mathbb{P}^N$ be set where $x_i \neq 0$ in homogeneous coords $[x_0 : \dots : x_N] \in \mathbb{P}^N$.

$$\cancel{\text{Spec}}\ \mathbb{A}^N = \operatorname{Spec} \mathbb{C}\left[\tfrac{x_0}{x_i}, \dots, \tfrac{x_{i-1}}{x_i}, \tfrac{x_{i+1}}{x_i}, \dots, \tfrac{x_N}{x_i}\right] \longleftarrow X_i$$

$$\mathbb{C}\left[\tfrac{x_0}{x_i}, \dots, \tfrac{x_N}{x_i}\right] \longrightarrow \mathcal{O}_{X_i}$$

$$\frac{x_j}{x_i} \longmapsto \frac{s_j}{s_i}.$$

This gives a map

$$X = \bigcup X_i \longrightarrow \bigcup \mathbb{A}^N = \mathbb{P}^N.$$

How to prove Bondal-Orlov?

Sketch: (1) Characterize skyscraper sheaves using categorical/algebraic properties

(2) "Construct"/topologize moduli space of skyscrapers using invertible sheaves.

Recall: If X smooth, $\dim_{\mathbb{C}} X = n$, then $\exists$ a dualizing sheaf ω_X s.t.

$$\mathrm{Ext}^i(A,B) \cong \mathrm{Ext}^{n-i}(B, A\otimes\omega_X)^\vee.$$

ω_X is the line bundle of holomorphic top forms, a.k.a. canonical bundle.

Defn Let D be a $\mathbb{C}$-linear category s.t. Ext^i are all finite-dimensional. A Serre functor for D is a functor

$$S: D \to D$$

s.t.

(1) S is a categorical equivalence of $\mathbb{C}$-linear categories

(2) We are given functorial isomorphisms

$$\hom_D(A,B) \cong \hom_D(B,SA)^\vee,$$

making

$$\begin{tikzcd} \hom_D(A,B) \arrow[r, phantom, "\cong"] \arrow[d, "S"', "\wr"] & \hom_D(B,SA)^\vee \\ \hom_D(SA,SB) \arrow[r, phantom, "\cong"] & \hom_D(SB,S^2A)^\vee \arrow[u, "S^{-1}"', "\wr"] \end{tikzcd}$$

commute.

~~Thm Let D be graded, so it has data of an autoequivalence $T: D \to D$, called the translation functor.~~

Thm Serre functors are unique up to autoequivalence of D.

Rmk. The Serre functor is actually $\mathcal{F} \longmapsto \mathcal{F} \otimes \omega_X[n]$, $n = \dim_{\mathbb{C}} X$, to make grading work.

In Bondal-Orlov they must fix a translation functor $[1] : D \to D$, just to work w/ greater generality than triangulated categories.

We'll take $A[1] := \operatorname{hocolim}\begin{pmatrix} A & \to & 0 \\ \downarrow & & \\ 0 & & \end{pmatrix}$ or, if you like, the object in the triangle $A \to 0 \to A[1] \xrightarrow{+}$.

Defn An object $P \in D$ is called point-like if

(1) $S(P) \simeq P[\ell]$ for some $\ell \in \mathbb{Z}$. (S is Serre duality functor)

(2) $\operatorname{Hom}^{<0}(P, P) = 0$

(3) $\operatorname{Hom}^{0}(P, P) = k_P$, some field dependent on P.

Prop'n $P \in D = D^b\operatorname{Coh}(X)$ is a point object iff $P \cong \mathcal{O}_x$ (up to a shift). I.e., P is a skyscraper sheaf.

Pf. Let $P \to I^0 \to I^1 \to \cdots$ be injective resolution,

$0 \to \mathcal{H}om(\mathcal{O}, I^0) \to \mathcal{H}om(\mathcal{O}, I^1) \to \cdots$

be derived $\mathcal{H}om$ sheaf, and $\mathcal{H}^i$ the cohomology (sheaves).

Since $S(P) \simeq P[n] \Rightarrow P \otimes \omega_X[n] \simeq P[n]$

$\Rightarrow P \otimes \omega_X \simeq P$,

we conclude $\mathcal{H}^i \otimes \omega_X \simeq \mathcal{H}^i$.

Now, ω_X is ample, so some $\mathcal{L} = \omega_X^{\otimes N}$ is very ample.

(ie, $\mathcal{L} \simeq j^* \mathcal{O}(1)$ for the embedding defined by $\mathcal{L}$ being generated by global sections.)

The Hilbert polynomial satisfies

$$\chi(\mathcal{H}^i \otimes \mathcal{L}^n) = \underbrace{P(n)}_{\text{Hilb. polyn. for } \mathcal{H}^i \text{ wrt } \mathcal{L}}.$$

But $\mathcal{H}^i \otimes \mathcal{L}^n = \mathcal{H}^i$, so $P(\cdot)$ is a <u>constant</u> polynomial!

Since degree of Hilb. polynomial = dimension of support of sheaf,

$\mathcal{H}^i$ is supported along points.

Now we must show • $\mathcal{H}^i$ has no nilpotence (so it doesn't look like $\mathbb{C}[x]/x^N$, which is also supported @ a point)

• $\mathcal{H}^\bullet$ is concentrated in a single degree w/ rank 1.

This is done using Grothendieck-Leray spectral sequence

$$Ext^\bullet(\mathcal{H}^i, \mathcal{H}^j) \Longrightarrow Ext(P, P),$$

citing (2) and (3).

Candidates for stability on A model

Defn X is called a Calabi-Yau manifold if

- X is Kähler w/ Kähler class $\omega^{1,1}$
- X is equipped w/ a trivialization of Ω^{top}
  (ie, a holomorphic $(n,0)$-form $\Omega^{n,0}$ which is nowhere vanishing)
- $(\omega^{1,1})^{top} = (-1)^{\text{something}} \Omega^{n,0} \wedge \overline{\Omega^{n,0}}$.

Recall that the usual statement of HMS (which I generalized last time by waving my hands) says

Conj $\forall X$ CY, $\exists X^\vee$ CY s.t.

$$Fuk(X) \simeq D^bCoh(X^\vee)$$
$$D^bCoh(X) \simeq Fuk(X^\vee).$$

If you believe this, stability conditions on $D^bCoh(X)$ should give rise to stability conditions on $Fuk(X^\vee)$. What are they? __Claim__: Determined by $\Omega^{3,0}$.

Let $L \subset X$ be a oriented Lagrangian. Lagrangian means $\dim_{\mathbb{R}} L = \frac{1}{2}\dim_{\mathbb{R}} X$, and $\omega^{1,1}|_L \equiv 0$.

Since L is oriented, $\Omega^{3,0}|_L$ determines some top form, $\mathbb{C}$-valued. So

$$\Omega^{3,0}|_L = e^{i\phi(x)} \, Vol$$

where $\phi(x) \in \mathbb{R}$ depends on $x \in L$. We say L is __special__ if ϕ is constant.

Now let $L \subset X$ be a Lagrangian, not necessarily special, but oriented

Since $\Omega^{3,0}|_L$ is nowhere vanishing, it makes sense to define a function

$$\mathrm{Arg}(\Omega^{3,0}|_L) : L \longrightarrow S^1 = \mathbb{C}^\times / \mathbb{R}$$

since ambiguity of L's framing at a point only changes scale.

We assume L is Maslov zero which means $\exists$ lift

$$\begin{tikzcd} & \mathbb{R} \arrow[d] \\ \mathrm{Arg}(\Omega^{3,0}|_L) : L \arrow[r] \arrow[ur] & S^1 \end{tikzcd} .$$

Idea (Joyce, Thomas, Yau, ...)

In space of all Lagrangians w/ Maslov zero, examine the function

$$\{L\} \longrightarrow \mathbb{R}$$

$$L \longmapsto \int_L |d\,\mathrm{Arg}\,\Omega^{3,0}|_L|^2$$

ie, the L^2 norm of $d\,\mathrm{Arg}\,\Omega^{3,0}|_L$.

Critical points are where $\Delta_L \mathrm{Arg}\,\Omega^{3,0}|_L = 0$;
that L is Maslov zero turns out to imply that Arg must hence be constant.

$\Longrightarrow$ Critical points are special Lagrangians!

In general, if you compactify $\{L\}$, so that gradient flow reaches a critical point, a Lagrangian L will flow NOT to a single special Lagrangian but to some singular Lagrangian

$$L \xrightarrow{\ \nabla \text{ flow}\ }$$

with $\lim L$ looking like a reducible Lagrangian w/ irreducible components *special* Lagrangian L_i, with

$$\operatorname{Arg}(\Omega^{3,0}|_{L_i}) > \operatorname{Arg}(\Omega^{3,0}|_{L_{i+1}}).$$

So there should be a "flow" called mean curvature flow which takes an object L to its H-N filtration!