10 September 2013

Last time:

<u>Thm</u> (Toda '12) Let X be a projective smooth variety of $\dim_{\mathbb{C}} X = 2$. Define

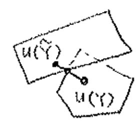
$$\begin{tikzcd} \mathrm{Stab}_{\mathbb{R}}\,\mathcal{C} \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & \mathrm{Stab}\,\mathcal{C} \arrow[d] \\ NS_{\mathbb{R}} \arrow[r] & G \end{tikzcd}$$

For all Y with a birational map $X \dashrightarrow Y$, there exists $U \subset \mathrm{Stab}_{\mathbb{R}}\mathcal{C}$ such that

(1) $U(Y)$ is open and nonempty

(2) $\mathcal{M}_\sigma([\mathcal{O}_X]) \cong Y \quad \forall\, \sigma \in U(Y)$

(3) If $X \dashrightarrow Y$ factors through $X \dashrightarrow \hat{Y} \to Y$, $\hat{Y}$ is the blowup of Y at a point, then $\overline{U(Y)} \cap \overline{U(\hat{Y})} \neq \phi$.



<u>Rmk</u> (1) Often, $\dim \mathrm{Stab}^{\text{Bridgeland}}(\mathcal{C}) = 2 \dim \mathrm{Stab}^{\text{Physics}}(\mathcal{C})$. Here, Toda picked out something of the correct "physical" dimension

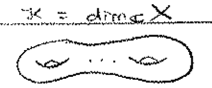
(Hiro doesn't know the definition, ask Kontsevich.)
((Murad says, you expect space of deformations of certain theories.))

(2) X need not be Calabi-Yau.

<u>Kodaira Dimension</u>

To each projective variety X, we can associate a number $\kappa(X) \in \{-\infty\} \cup \{0, 1, \cdots, \dim X\}$, called the <u>Kodaira dimension</u> of X.

more generic →

$\dim X$	$\kappa = -\infty$	$\kappa = 0$	$\cdots$	$\kappa = \dim_{\mathbb{C}} X$
1				 genus ≥ 2
$n \geq 2$	• X birational to $\mathbb{P}^n$ $\cap$ del Pezzo surfaces, Fano varieties K_X^{-1} ample (if you know bounded derived category of coherent sheaves, can recover X)	• Abelian varieties • Calabi-Yau varieties	?	• "general type" <u>Ex</u> $X \subset \mathbb{P}^N$ hypersurface of degree $\geq N+1$ <u>Ex</u> K_X ample

the bundle of holomorphic top forms

<u>Def</u> $\bigoplus_{n\geq 0} H^0(K_X^{\otimes n})$ is a graded ring, called the "<u>canonical ring</u>" of X.

$\mathrm{Proj}\left(\bigoplus_{n\geq 0} H^0(K_X^{\otimes n})\right)$ is a variety of dimension $\kappa(X)$, the Kodaira dimension.

<u>Exer</u> If $K_X \cong \mathcal{O}_X$, show $\kappa(X) = 0$.

(Hint: • X projective, so $H^0(\mathcal{O}_X) \cong \mathbb{C}$.

• $\mathcal{O}_X$ is unit w.r.t. tensor product.)

<u>Proof</u> $\bigoplus_{n\geq 0} H^0(K_X^{\otimes n}) \cong \mathbb{C}[x]$

$\mathrm{Proj}(\mathbb{C}[x]) \cong \mathbb{CP}^0 = \{*\}$

Goal of Minimal Model Program (for varieties of general type)

Given X of general type, find X' such that

- $X \dashrightarrow X'$ birational
- for all curves $C \subset X'$, $K_{X'} \cdot C > 0$

Thm (Castelnuovo) If X is a surface, this can be achieved by contractions of $\mathbb{P}^1 \subset X$.

Reconstruction

Toda's theorem says that there exists $U \in \mathrm{Stab}_{\mathbb{R}}\, \mathcal{C}$ where $M_\sigma([\mathcal{O}_x]) \cong X \quad \forall \sigma \in U$

Q Can you reconstruct X from $\mathcal{C} = D^b \mathrm{Coh}(X)$?

A (Toda) If X is a surface, yes if you choose correct stability condition.

Thm (Bondal-Orlov) If X has ample K_X or ample K_X^{-1}, you can reconstruct X from $D^b\mathrm{Coh}(X)$

(all you need is the choice of a shift functor $[1]$).

Proof involves two ideas:

(1) Pick out "skyscraper sheaves" using algebraic/categorical criteria.

(2) "Topologize" the set of skyscraper-like objects (using "line bundle" objects)

$$\mathcal{C} \rightsquigarrow \{\text{point-like objects}\}$$
$$\rightsquigarrow \{\text{line bundle objects}\}$$

Recall: for all smooth X, $\dim_{\mathbb{C}} X = n$, there exists a dualizing sheaf ω_X:

$$\mathrm{Ext}^i(A, B) \cong \mathrm{Ext}^{n-i}(B, A \otimes \underbrace{\omega_X}_{\text{canonical bundle}})^\vee \qquad \text{(Serre duality)}$$

Def Let D be $\mathbb{C}$-linear category such that Ext^i are all finite-dimensional. A Serre functor for D is $S: D \longrightarrow D$ such that

(1) S is an equivalence of $\mathbb{C}$-linear categories

(2) We're given functorial isomorphisms $\hom_D(A, B) \cong \hom_D(B, SA)^\vee$

(so $S = - \otimes \omega_X[n]$ for $D^b\mathrm{Coh}(X)$)

(3) The diagram

$$\begin{array}{ccc} \hom(A,B) & \xrightarrow{\cong} & \hom(B, SA)^\vee \\ \cong\downarrow & & \uparrow\cong \\ \hom(SA, SB) & \xrightarrow{\cong} & \hom(SB, S^2A)^\vee \end{array}$$

commutes.

Thm If S exists, it is unique up to automorphisms of D.

Def An object $P \in D$ is called point-like if

(1) $S(P) \cong P[\ell]$ for some $\ell \in \mathbb{Z}$ (think about what conditions skyscraper sheaf should satisfy)

(2) $\mathrm{Hom}^{<0}(P, P) = 0$

(3) $\mathrm{Hom}^0(P, P)$ is a field.

Thm If P is point-like, then $P \cong \mathcal{O}_x[l] \in D^b\mathrm{Coh}(X)$ for some $l \in \mathbb{Z}$. (Note: ampleness is crucial!)

Proof Let $P \longrightarrow I^0 \longrightarrow I^1 \longrightarrow \cdots$ be an injective resolution

Sheaf Hom $\quad 0 \longrightarrow \mathcal{H}om(\mathcal{O}, I^0) \longrightarrow \mathcal{H}om(\mathcal{O}, I^1) \longrightarrow \cdots$

Let $\mathcal{H}^i$ be cohomology sheaves of P.

Assume $l = n$.

By (1), $S(P) \cong P[n]$

$\Longrightarrow P \otimes \omega_X[n] \cong P[n]$

$P \otimes \omega_X \cong P$

$\Longrightarrow \mathcal{H}^i \otimes \omega_X \cong \mathcal{H}^i$

Examine Hilbert polynomial of $\mathcal{H}^i$. (degree of Hilbert polynomial $= \dim(\operatorname{supp} \mathcal{H}^i)$)

The Hilbert polynomial $P_{\mathcal{H}}$ w.r.t. ω_X satisfies $\chi(\mathcal{H}^i \otimes \omega_X^n) = P_{\mathcal{H}}(n)$.

But $\mathcal{H}^i \otimes \omega_X^n = \mathcal{H}^i$, so $P_{\mathcal{H}}$ is constant $\Longrightarrow \operatorname{supp}(\mathcal{H}^i)$ is some union of points

Use spectral sequence $\mathrm{Ext}^{\circ}(\mathcal{H}^i, \mathcal{H}^i) \Longrightarrow \mathrm{Ext}(P,P)$ and (2), (3) to conclude P is a shift of $\mathcal{O}_x$.

Stability Conditions on A-Model

Homological Mirror Symmetry conjecture:

A-model (symplectic)		B-model
$\mathrm{Fukaya}(X)$	$\cong$	$D^b\mathrm{Coh}(X)$
$D^b\mathrm{Coh}(X)$	$\cong$	$\mathrm{Fukaya}(X^\vee)$

Def X is called a Calabi-Yau manifold if

- X is Kähler, $\omega^{1,1}$ Kähler form
- X is equipped with a trivialisation $\Omega^{top} \cong \mathcal{O}_X$
- $(\omega^{1,1})^{top} = (-1)^{sign}\, \Omega^{n,0} \wedge \overline{\Omega}^{n,0}$ $\qquad$ ($\Omega^{n,0}$ nowhere vanishing section)

Claim If X is a Calabi-Yau 3-fold, $\Omega^{3,0}$ picks out stable Lagrangians.
(= special)

Def $L \subset X$ is Lagrangian if

- $\dim_{\mathbb{R}} L = \frac{1}{2} \dim_{\mathbb{R}} X$
- $\omega^{1,1}|_L \equiv 0$

If L is oriented, $\Omega^{3,0}|_L$ defines a $\mathbb{C}$ valued volume form $\Omega^{3,0}|_L = e^{i\phi(x)} \cdot \mathrm{volume}$. $\phi(x)$ depends on $x \in L$.

Def L is special if $\phi(x)$ is constant on L.