Bridgeland Stability Conditions

09/10/13 (Lecture 3)

1 A Theorem of Toda and the MMP

(For further information, see Toda's paper at arxiv:1205.3602.)

1.1 Review from Last Time

For this discussion, let X be a fixed complex projective smooth variety of dimension 2. Let $\mathrm{Stab}(X)$ be the moduli space of Bridgeland stability conditions on $D^b\mathrm{Coh}(X)$, organized into a complex manifold (note, we still haven't defined this topology, we haven't defined what it means for two $\heartsuit$'s to be close), and let $\mathrm{Stab}(X)_{\mathbb{R}}$ be the 'real' part of this manifold. Note the following pullback square

$$\begin{array}{ccc} \mathrm{Stab}(X)_{\mathbb{R}} & \longrightarrow & \mathrm{Stab}(X) \\ \downarrow & & \downarrow \\ \mathrm{NS}(X)_{\mathbb{R}} & \longrightarrow & N(X)^{\vee}_{\mathbb{C}} \end{array}$$

Here $\mathrm{NS}(X)$ refers to the Neron-Severi group, and $N(X)$ the numerical Grothendieck group of X. For a fixed Bridgeland stability condition $\sigma = (Z, \mathcal{C})$, denote by $\mathcal{M}^{\sigma}([\mathcal{O}_x])$ the (algebraic) moduli space of Z-stable objects in $\mathcal{C}$ with $\phi = 1$ and $\mathrm{ch}(E) = \mathrm{ch}(\mathcal{O}_x)$ for $x \in X$.

Theorem 1.1. *(Toda, 2012) For all smooth projective Y with a birational map $X \to Y$, there is an open, nonempty subset $U(Y) \subset \mathit{Stab}(X)_{\mathbb{R}}$ such that*

1. *$\mathcal{M}^{\sigma}([\mathcal{O}_x]) \simeq Y$ for all $\sigma \in U(Y)$.*
2. *If $X \to Y$ factors through $\tilde{Y} \to Y$ (the blowup of Y at a point), then $\overline{U(Y)} \cap \overline{U(\tilde{Y})}$.*

The rough intuition of this second condition is that, in the space of stability conditions, we can 'move' along from one $U(Y)$ to the next, along a sequence of birational equivalences (in particular, to its minimal model).

Note that, here, X need not be Calabi-Yau! Any dimension 2 smooth complex projective variety will do.

Often, $\mathrm{dimStab}^{\mathrm{BL}}(X) = 2\mathrm{dimStab}^{\mathrm{physics}}(X)$. Here, the moduli space of 'physics' stability conditions refers to the space of deformations (?). In general, the number of objects generating the Grothendieck group should be double the dimension of the deformation space. (? I was a bit confused here with these comments.)

Last, note that by the special case where $Y = X$, we have an open subset $U \subset \mathrm{Stab}(X)_{\mathbb{R}}$ such that $\mathcal{M}^{\sigma}([\mathcal{O}_X]) = XL$ for all $\sigma \in U$. This will be important when we revisit the reconstruction problem.

1.2 Kodaira Dimension

The Kodaira dimension $K(X)$ of a projective variety X is an invariant constructed from the canonical bundle that roughly measures how 'generic' X is among projective varieties of that dimension.

Definition 1.2. *Let $K_X = \Omega_X^d$ denote the line bundle of holomorphic top forms (d is the dimension of X). Then we can construct the graded ring $S(X) = \bigoplus_{n\geq 0} H^0(X, K_X^{\otimes n})$. The ring structure arises in the obvious way. This is called the 'canonical ring' of X. $K(X)$ is simply the dimension of the projective variety Proj$(S(X))$: it is either a nonnegative integer in $\{0, 1, \ldots, dimX\}$, or $-\infty$.*

(Question by Saul: What if we take $\bigoplus_{n\geq 0} H^{\bullet}(K_X^{\otimes n})$? This has a ring structure as well, because elements of Ext$^{\bullet}(\mathcal{O}_X, K_X^{\otimes n})$ and Ext$^{\bullet}(\mathcal{O}_X, K_X^{\otimes m})$ can be composed by pushouts. What happens?)

Exercise: If $K_X = \mathcal{O}_X$, prove that $K(X) = 0$.

Solution: $S(X) = \bigoplus_{n\geq 0} H^0(\mathcal{O}_X^{\otimes n}) \simeq \bigoplus_{n\geq 0} H^0(\mathcal{O}_X) \simeq \bigoplus_{n\geq 0} \mathbb{C} \simeq \mathbb{C}[x]$

Now let's write some examples of projective varieties and their Kodaira dimensions.

<u>$\dim(X) = 1$</u>
$K = -\infty$: $\mathbf{CP}^1$
$K = 0$: T^2 (the torus)
$K = 1$: Any surfaces of genus ≥ 2

<u>$\dim(X) \geq 2$</u>

$K = -\infty$: X birational to $\mathbf{P}^n$ works. In general, del Pezzo surfaces, or Fano schemes (i.e., where K_X^{-1}, the inverse of the canonical line bundle under tensor product, is ample) work.

$K = 0$: Abelian varieties, or Calabi-Yau varieties.

$\vdots$

$K = \dim(X)$: This is the 'generic' case. For example, if X is a hypersurface in some $\mathbb{P}^N$ of degree $> N + 1$.

1.3 Goal of the Minimal Model Program (for varieties of general type)

Given X of general type (i.e., $K(X) = \dim(X)$?), we'd like to find some X' (a 'minimal model') such that

- There is a birational map $X \to X'$
- For all curves $C \subset X'$, $K_{X'} \cdot X \geq 0$.

Theorem 1.3. *(Castelnuovo) If X is a surface, then this can be achieved by successive contractions of $\mathbb{P}^1$'s in X (i.e., blow-downs).*

2 The Reconstruction Problem

2.1 The Idea

Recall the reconstruction problem, where, given the triangulated category $D^b\text{Coh}(X)$ and possibly some other information, we want to reconstruct the variety X. Remember now that by Toda's theorem, applied in the special case where $Y = X$, we have an open subset $U \subset \text{Stab}(X)_{\mathbb{R}}$ such that $\mathcal{M}^\sigma([\mathcal{O}_X]) = X$ for all $\sigma \in U$. So somehow, this suggests a way to recover X as the moduli space of objects stable under a particular class U of stability conditions.

So, when can we reconstruct X? It turns out that if X is a surface, you can if you choose the right stability conditions.

Theorem 2.1. *(Bondal-Orlov) If X has ample K_X or K_X^{-1}, you can reconstruct X from D^bCoh(X), if you have the shift functor* [1].

The proof of this theorem involves two key ideas.

1. Is there a set of objects which seem like they could allow you to reconstruct X? Well it would be really nice if we could get ahold of the skyscraper sheaves on X. So using algebraic-categorical criteria, namely, the notion of a *point object*, we'll be able to pull these objects out.
2. We also need some way to 'topologize' this set of point objects, so that we know how to glue them together into X. In order to do so, we define other algebraic criteria to pick out objects that look like line bundles on X. The idea is that if we know the line bundles, we can define their divisors, and the complement of a divisor is an open set on the collection of point objects. This allows us to recover the topology.

2.2 Algebraic Nonsense

Recall that if X is a smooth projective variety over $\mathbb{C}$ of dimension n, there is a *dualizing sheaf* ω_X (also called the *canonical bundle*) such that

$$\text{Ext}^i(A, B) \simeq \text{Ext}^{n-i}(B, A \otimes \omega_X)^\vee$$

(This is the statement of Serre duality.)

Definition 2.2. *Let $\mathcal{D}$ be a $\mathbb{C}$-linear category. Suppose that all Ext^i's are finite dimensional in $\mathcal{D}$. A* **Serre functor** *$S : \mathcal{D} \to \mathcal{D}$ is an additive functor such that*

- *S is an equivalence of categories.*
- *There are functorial isomorphisms $hom_{\mathcal{D}}(A, B) \simeq hom_{\mathcal{D}}(B, SA)^\vee$. (So for example, if $\mathcal{D} = D^b Coh(X)$, $S(-) = - \otimes \omega_X[n]$.)*
- *The following diagram commutes*

$$\begin{array}{ccc} hom(A,B) & \xrightarrow{iso} & hom(B,SA)^\vee \\ \big\downarrow S & & \big\uparrow S^\vee \\ hom(SA,SB) & \xrightarrow{iso} & hom(SB,S^2A)^\vee \end{array}$$

  where all arrows are isomorphisms.

Theorem 2.3. *If S exists, it's unique up to automorphisms of $\mathcal{D}$.*

Definition 2.4. *An object $P \in ob\mathcal{D}$ is* **point-like** *if*

1. *$S(P) = P[\ell]$ for some integer ℓ. (In $D^b Coh(X)$, $S(P) = P \otimes \omega_X[n] = P[n]$.)*
2. *$Hom^{<0}(P, P) = 0$ (these Hom sets are graded $\mathbb{C}$-algebras)*
3. *$Hom^0(P, P)$ is a field.*

(Remark: You can compute that Ext of a skyscraper sheaf is a torus. Well, it turns out that Lagrangian tori in the A-model correspond to skyscraper sheaves in the B-model.)

Theorem 2.5. *If P is point-like and $\mathcal{D} = D^b Coh(X)$, then $P \simeq \mathcal{O}_X[\ell]$ for some ℓ.*

(Alert: the ampleness of P is crucial here, because it allows you to build maps to some $\mathbb{P}^n$.)

Proof. We'll mention a piece of the proof. Let $P \to I^\bullet$ be an injective resolution. Then we have a sequence of sheaves

$$0 \to \mathscr{H}(\mathcal{O}_X, I^0) \to \mathscr{H}(\mathcal{O}_X, I^1) \to \cdots$$

Let $\mathscr{H}^i$ be the cohomology sheaves of P (i.e., the cohomology sheaves of this sequence). By (1) in the definition of point-like, $S(P) \simeq P[n]$ (assume $\ell = n$). Then

$$P \otimes \omega_X[n] \simeq P[n] \implies P \otimes \omega_X \simeq P$$
$$\implies \mathscr{H}^i \otimes \omega_X \simeq \mathscr{H}^i$$

So now let $P_{\mathscr{H}^i}$ be the Hilbert polynomial of $\mathscr{H}^i$. (remember that the degree is $\dim(\mathrm{supp}(\mathscr{H}^i))$.)

$$\chi(\mathscr{H}^i) = \chi(\mathscr{H}^i \otimes \omega_X^{\otimes m}) = P_{\mathscr{H}^i}(m)$$

for any m. So $P_{\mathscr{H}^i}$ is constant, and thus $\mathrm{supp}(\mathscr{H}^i)$ is a union of closed points.

(There's still work to do. We need to use the spectral sequence

$$\mathrm{Ext}^\bullet(\mathscr{H}^i, \mathscr{H}^j) \implies \mathrm{Ext}^\bullet(P, P)$$

and then utilize properties (2) and (3) of pointlike objects to get the result. We leave this computation out.) □

3 Stability Conditions on the A-Model

First off, I'll mention a mnemonic to help you remember what the A-model and B-model are. The homological mirror symmetry conjecture states that you have equivalences of categories

$$\mathrm{Fukaya}(X) \longleftrightarrow D^b\mathrm{Coh}(X^\vee)$$

The a's in Fukaya indicate that it is the A-model, and the b in $D^{\boxed{b}}$Coh indicate that it is the B-model. Now that that's out of the way...

Definition 3.1. *X is called Calabi-Yau if*

- *X is Kahler ($\omega^{1,1}$ Kahler form)*
- *X is equipped with a trivialization $\Omega^n \simeq \mathcal{O}_X$ (a nowhere vanishing section of $\Omega^{n,0}$, n being the dimension of X)*
- $(\omega^{1,1})^n = (-1)^{sign}\Omega^{n,0} \wedge \overline{\Omega^{n,0}}$

We claim that if X is a Calabi-Yau 3-fold, then $\Omega^{3,0}$ picks out stable Lagrangians.

3.1 Setup for nenxt time

Definition 3.2. *$L \subset X$ is* Lagrangian *if*

- $dim_{\mathbb{R}} L = \frac{1}{2} dim_{\mathbb{R}}(X)$
- $\omega^{1,1}|_L = 0$

If L is oriented, $\Omega^{3,0}|_L$ defines a $\mathbb{C}$-valued voume form. That is, $\Omega^{3,0}|_L$ equals $e^{i\phi(x)}$ times the volume form. What happens if we integrate this? We say L is a *special Lagrangian* if ϕ is constant on L.