12 September 2013

From last time. Conjectural stability conditions for the A model

Idea: $D^b \mathrm{Coh}(X) \simeq \mathrm{Fuk}(X^\vee)$
symplectic invariant of $X^\vee$
(doesn't depend on choice of complex structure)

What is Fuk? Objects: $L \subset X^\vee$ Lagrangians ($\dim_{\mathbb{R}} L = \frac{1}{2}\dim_{\mathbb{R}} X^\vee$, $\omega|_L \equiv 0$)

(Ex • $X^\vee = \mathbb{R}^2$ $dx \wedge dy$, any curve is a Lagrangian

• real locus of a complex projective variety, e.g. $\mathbb{RP}^n \subset \mathbb{CP}^n$)

Morphisms: $\mathrm{Hom}(L_0, L_1)$ is a chain complex (coefficients are tricky, for the time being think $\mathbb{Z}$)

• generated by intersection points $L_0 \cap L_1$.

Ex T^*Q zero section is a Lagrangian L_0

graph (df), $f: Q \longrightarrow \mathbb{R}$ is also Lagrangian L_1

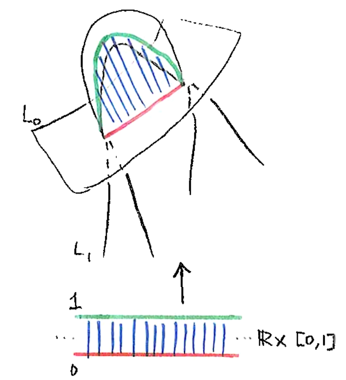
$L_0 \cap L_1 = \mathrm{Crit}(f)$

• Differential given by counting J-holomorphic strips
$(du \circ J_{\mathbb{C}} = J_{X^\vee} \circ du)$

$$\mathbb{R} \times [0,1] \xrightarrow{u} X^\vee$$

gives complex structure $\dashrightarrow$ $\cap\ \mathbb{C}$

With appropriate boundary conditions



Q What are the stable objects of Fuk?

(As of a couple of months ago there are stability conditions on abelian 3-folds)

Recall: if $X^\vee$ is Calabi-Yau, we have $\Omega^{n,0}$ a holomorphic n-form.

Let $\dim_{\mathbb{C}} X^\vee = 3$ $(\Omega^{3,0})$ If $L \subset X^\vee$ is oriented, $\Omega^{3,0}|_L = e^{i\phi(x)} \mathrm{vol}_L$, $\phi(x) \in \mathbb{R}$ depends on $x \in L$

Def L is called special if ϕ is constant

Idea: Z: ob Fuk $\longrightarrow \mathbb{C}$ should be $\int_L \Omega^{3,0}|_L$, special Lagrangians should be stable objects.

Q How do you get Harder Narasimhan filtration for L?

For all Lagrangians L which are oriented, there are functions $\mathrm{Arg}(\Omega^{3,0}|_L) : L \longrightarrow S^1$.

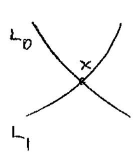
Assume L is Maslov zero, i.e. there exists a lift
$$\begin{tikzcd} & \mathbb{R} \arrow[d] \\ L \arrow[ur, "f"] \arrow[r] & S^1 \end{tikzcd}$$

$X^\vee = \mathbb{R}^{2n} \cong \mathbb{C}^n$

$\mathrm{Gr\,Lag}(\mathbb{R}^{2n}) = U(n)/O(n)$

$U(n)/O(n) \overset{\det^2}{\dashrightarrow} S^1$

$$\begin{tikzcd} & \mathrm{Gr\,Lag} \arrow[r, "\det^2"] \arrow[d] & S^1 \\ L \arrow[ur] \arrow[r, hook] & X^\vee & \end{tikzcd}$$
if X is CY



$$\begin{tikzcd} & \mathbb{R} \arrow[d] \\ x \arrow[ur, "f_0"] \arrow[r, "f_1"'] & \mathbb{R} \end{tikzcd}$$

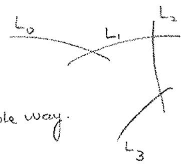
<u>Fairy tale</u> (Yau, Smith, Joyce) Look at $\{L\}$ = space of Maslov zero Lagrangians.

$$\{L\} \longrightarrow \mathbb{R}$$

$$L \longmapsto \int_L |\, d\mathrm{Arg}(\Omega^{3,0})|^2$$

Critical points $\Delta_L \mathrm{Arg}(\Omega^{3,0})|_L \equiv 0.$

Maslov zero $\Longrightarrow$ Arg constant

In general, L should flow to some singular Lagrangian L', L' = 

L_i special Lagrangians

We should also be able to say $\phi(L_0) \supset \phi(L_1) \supset \cdots$ in some sensible way.