Bridgeland Stability Conditions

09/12/13 (Lecture 4)

1 (Conjectural) Stability Conditions for the A-Model

Recall the Homological Mirror Symmetry Conjecture states that there is an equivalence of categories between $D^b\mathrm{Coh}(X)$ and $\mathrm{Fuk}(X^\vee)$. Let's actually define what the Fukaya category is.

1.1 Fuk(X)

Objects: The objects of $\mathrm{Fuk}(X)$ will be Lagrangian submanifolds of X. That is, they will be submanifolds $L \subset X$ with $\dim \mathbb{R} L = \frac{1}{2}\dim_{\mathbb{R}} L$, and $\omega|_L = 0$.

For example, suppose $X = \mathbb{R}^2$, and $\omega = dx \wedge dy$. Then any curve is a Lagrangian. Another example would be in the case where X is a complex projective variety. Then the real locus of X is one such Lagrangian (for example, $\mathbf{RP}^n \subset \mathbf{CP}^n$).

Morphisms: For Lagrangians L_0 and L_1, $\mathrm{Hom}(L_0, L_1)$ will be a chain complex: for now, let's just say a chain complex of abelian groups. This chain complex will be 'generated' by the points where L_0 and L_1 intersect (the dimension of $L_0 \cap L_1$ will be 0, clearly), where different points have different degree depending on local geometric structure (which we'll define later). We'll explore an example to demonstrate what's going on. (I only roughly understood this construction, so corrections are appreciated)

Let $X = T^*Q$ be the cotangent bundle of some smooth Q. Then the zero section is a Lagrangian: call it L_0. If $f : Q \to \mathbb{R}$ is a smooth function, then $\mathrm{Graph}(df) \subset T^*Q$ is a section which is also Lagrangian. $L_0 \cap L_1$ is precisely the set of points where $df = 0$, namely, the critical points of f.

The differential map in our chain complex is given as follows. Given two critical points p and q, we count J-holomorphic strips $H : \mathbb{R} \times [0,1] \to X$ (where $\mathbb{R} \times [0,1]$ is thought of as a subset of $\mathbb{C}$), subject to the following boundary conditions:

- $H(t,i)$ lands in L_i for $i = 0,1$.
- $H(0,s) = p$ and $H(1,s) = q$ for $s \in [0,1]$.

Here, 'J-holomorphic' means that $du \cdot J_{\mathbb{C}} = J_X \cdot du$.

1.2 Stability conditions

(I should note that this portion confused me, so if you find errors in the notes or things I have written poorly, please let me know!)

What are the stable objects of Fuk(X)**?** (Comment by Matt: As of a couple of months ago, there are known stability conditions for abelian 3-folds.) Recall that if X is Calabi-Yau, when we have a holomorphic n-form $\Omega^{n,0}$ (where $n = \dim_{\mathbb{C}} X$). Let's take $n = 3$ for now.

If $L \subset X$ is an oriented Lagrangian,

$$\Omega^{3,0}|_L = e^{i\phi(x)} \mathrm{vol}_L$$

L is called *special* if ϕ is constant over L. The idea here is that $Z : \mathrm{ob}(\mathrm{Fuk}(X)) \to \mathbb{C}$ should be integrating $\Omega^{3,0}$ over L. The special Lagrangians are then the stable objects for Z.

How do you get the Harder-Narasimhan filtration for L**?** We'll introduce some topogy to address this question. For an oriented Lagrangian L, we can define a function

$$\mathrm{Arg}(\Omega^{3,0}|_L) : L \to S^1$$

Assume L is *Maslov zero*, i.e., that this map lifts to $\mathbb{R}$ (and factors through the quotient $\mathbb{R} \to S^1$).

Let's consider the example $X \simeq \mathbb{R}^{2n} \simeq \mathbb{C}^n$. Then $\mathrm{GrLag}(X)$, the Grassmannian of Lagrangians in X is a $(U(n)/O(n))$-bundle over X. $\det^2$:

$(U(n)/O(n)) \to S^1$ is a well-defined map. So if X is Calabi-Yau, we have the following diagram

$$\begin{array}{ccccc} L & \longrightarrow & \mathrm{GrLag}(X) & \longrightarrow & S^1 \\ & \searrow & \downarrow & & \\ & & X & & \end{array}$$

Here, the vertical map is a $(U(n)/O(n))$-bundle, and in the first horizontal map, each L is just sent to itself in the Grassmannian. The second horizontal map exists if X is Calabi-Yau. This actually gives the geometric data we can use to determine what degree the points of $L_0 \cap L_1$ are at in the chain complex $\mathrm{Hom}(L_0, L_1)$. Suppose $x \in L_0 \cap L_1$. We can pick a lift of this map $\mathrm{GrLag}(X) \to \mathbb{R}$, and see where x is sent in the corresponding maps $f_0 : L_0 \to \mathbb{R}$ and $f_1 : L_1 \to \mathbb{R}$. Then the point x should show up in the chain complex in degree $f_1(x) - f_0(x)$.

In any case to get the Harder-Narasimhan filtration, there's a hopeful method by Yau, Smith, and Joyce. Look at the moduli space M of Maslov zero Lagrangians. Then we have a map $M \to \mathbb{R}$ which acts as

$$L \mapsto \int_L |d \ \ \mathrm{Arg}(\Omega^{3,0})|^2$$

The critical points are the Lagrangians L such that $\Delta_L \mathrm{Arg}(\Omega^{3,0})|_L = 0$, where Δ_L is the Laplacian. These should correspond to the special Lagrangians. The condition that L is Maslov zero corresponds to Arg being constant.

In general, L should 'flow' to a singular Lagrangian L' which is a union of special Lagrangians L_i. Then, we can perform surgery theory to get something smooth and Hamiltonian isotopic to L'. We should also be able to say $\phi(L_0) > \phi(L_1) > \cdots$ in some sensible way. So this successive 'flow' along a sequence of special Lagrangians (stable objects) should give our Harder-Narasimhan filtration.

Should we expect such a flow to give the Harder-Narasimhan filtration as well in the B-model? Kontsevich says 'Yes. You should expect it.'

(After the break, Murad took over and gave some of the physics side. He said he would provide notes from this talk.)