Murad Talk Two.

1) Into

1.1) Ingredients   4d, $N=2$

- $\mathcal{M}$ moduli space, $\dim \mathcal{M} = r$

  (r comes from Gauge group G, $U(1)^r$ max torus)

- Γ lattice of rank $2r$ $(\underbrace{e^\alpha}_{\text{electric}}, \underbrace{m_\alpha}_{\text{magnetic}}) = \gamma \in \Gamma$ — e.g., $H^{even}(X, \mathbb{Z})$, $H_3(X^\vee, \mathbb{Z})$

- pairing $\langle \, , \rangle : \Gamma \times \Gamma \to \mathbb{Z}$, symplectic

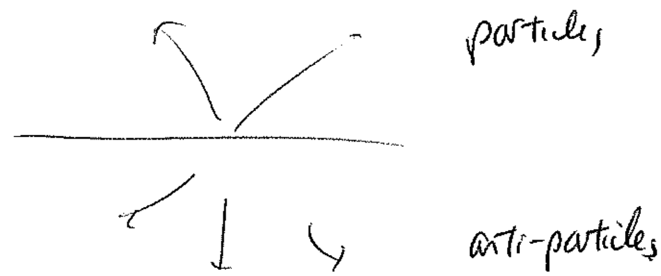
- $Z_u : \Gamma \longrightarrow \mathbb{C}$ u a parameter for moduli space.

  $\gamma \longmapsto Z_u(\gamma)$

  s.t. $\underbrace{M_u}_{\text{mass}}$ is constant by $|Z_u(\gamma)|$.

2) BPS Quiver

- $\gamma \in$ spectrum $\Rightarrow -\gamma \in$ spectrum.



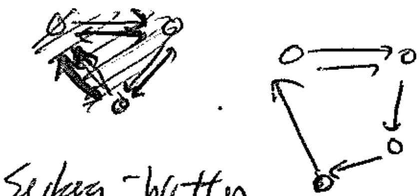
- pick "half" of spectrum using Z, called set of particles

basis $\{\gamma_1, \dots, \gamma_{2r}\}$ s.t. $\gamma \in$ particles

$$\gamma = \sum n_i \gamma_i \, , \; n \in \mathbb{N}.$$

<u>Quiver:</u> a node to every γ. Arrows $A_{ij}{}^{q}$, # arrows $= \langle \gamma_i, \gamma_j \rangle$

Quiver might have cycles, but not between just 2 nodes.



This characterizes 4-d SU(2) Seiberg-Witten

$\downarrow$

$$\circ^{\gamma_1} \rightrightarrows \circ^{\gamma_2}.$$

$\underset{\gamma_1}{\circ} \longrightarrow \underset{\gamma_2}{\circ}$ Does <u>Not</u> corresponds to a particular theory as of now

Why isn't every lattice point γ a BPS particle?

Given $\gamma = \sum_{i=1}^{2r} n_i \gamma_i$, is γ in the ~~spec~~ set of particles?

It'll turn out only "semistable" objects will be BPS states

3.1) To answer, let's get info out of quiver using quiver representations.

- $\mathbb{C}^{n_i}$ to node γ_i
- B_{ij}^{a} linear map $\mathbb{C}^{n_i} \to \mathbb{C}^{n_j}$.

$\Big\}$ γ_R a representation

# of vertices of quiver never changes. So you'll have at least as many BPS states as vertices.

$$\begin{tikzcd} \mathbb{C}^{n_i} \arrow[r, bend left, "B_{ij}^{a}"] \arrow[r] \arrow[r, bend right, "B_{ij}^{a_n}"'] & \mathbb{C}^{n_j} \end{tikzcd}.$$

A subrepresentation is $\gamma_S = \sum m_i \gamma_i$, $m_i \le n_i$ ← from γ_R.

s.t.

$$\begin{tikzcd} \mathbb{C}^{n_i} \arrow[r, "B_{ij}^{a}"] & \mathbb{C}^{n_j} \\ \mathbb{C}^{m_i} \arrow[u] \arrow[r, "B_{ij}^{a}"] & \mathbb{C}^{m_j} \arrow[u] \end{tikzcd}$$

Stability condition will see subrepresentations

Stability (Douglas, Fiol, Romelsberger 2000)

A representation γ_R is stable if $\forall$ subrepresentation γ_S,

$$\arg Z(\gamma_S) < \arg Z(\gamma_R).$$

Fixing n_i, which B_{ij}^a give rise to stable reps?

Define moduli space to be

$$\mathcal{M} = \{ B_{ij} : \mathbb{C}^{n_i} \to \mathbb{C}^{n_j} \quad \text{s.t.} \quad \gamma_R \text{ is stable} \} \Big/ \prod_i GL(n_i; \mathbb{C})$$

Ex $\begin{matrix} \gamma_1 & & \gamma_2 \\ \circ & \longrightarrow & \circ \end{matrix}$ A_2 Argyres-Douglas theory.

$\gamma = \gamma_1 + \gamma_2 \in$ particle.

$\mathbb{C} \to 0$
$0 \to \mathbb{C}$
are stable.

How about subreps of $\mathbb{C} \xrightarrow{B} \mathbb{C}$? (some #.)

- $\mathbb{C} \to 0$ is a sub if $B = 0$.

$$\begin{tikzcd} \mathbb{C} \arrow[r, "B"] & \mathbb{C} \\ \mathbb{C} \arrow[u] \arrow[r] & 0 \arrow[u] \end{tikzcd}$$

For this to be stable, we'll want

$$\arg Z(\gamma_1) < \arg Z(\gamma_1 + \gamma_2) < \arg Z(\gamma_2)$$

- $0 \to \mathbb{C}$ is always a sub, regardless of B

γ_R stable $\Rightarrow$ $\arg Z(\gamma_2) < \arg Z(\gamma_1)$.

$$\begin{tikzcd} \mathbb{C} \arrow[r, "B"] & \mathbb{C} \\ 0 \arrow[u] \arrow[r] & \mathbb{C} \arrow[u] \end{tikzcd}$$

So γ_R is stable only when $B \neq 0$ and $\arg Z(\gamma_2) < \arg Z(\gamma_1)$

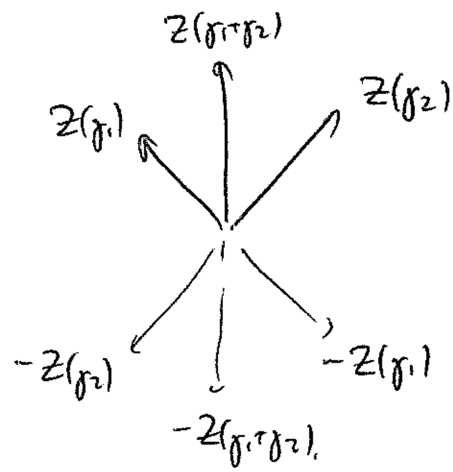
So

$$\mathfrak{M}_{\gamma_R} = \mathbb{C}^\times / \mathbb{C}^\times \times \mathbb{C}^\times . \quad \rightsquigarrow \quad * / \mathbb{C}^* \qquad \text{moduli stack}$$

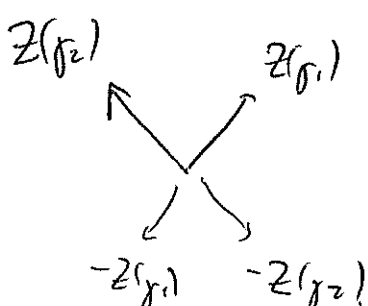
Sometimes, you mod out $\prod GL(n_i)$ by diagonal elements; just get moduli space.

$$\mathbb{C}^\times / \mathbb{C}^\times \quad \rightsquigarrow \quad * .$$

So for $\circ \rightarrow \circ$, we get



when $\arg Z(\gamma_1)$ approaches $\arg Z(\gamma_2)$, we may have decay



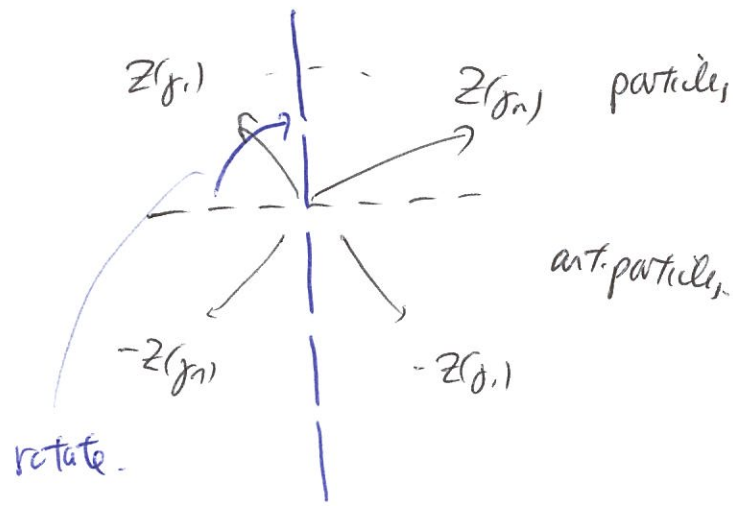
We call $Z(\gamma_1+\gamma_2)$ a bound <u>state</u>.

It decays into its constituents as you cross the wall.

Special Lagrangians of a curve, vary them to vary the physics (C structure deformations of a curve). More Bridgeland stability conditions than there are geometric deformations.

<u>Computing spectra</u>: If you know basis for BPS states, need to compute spaces of stable reps. Often there's a shortcut.

Given a choice of half the particles, we know boundary of the cone must contain two basis elements.



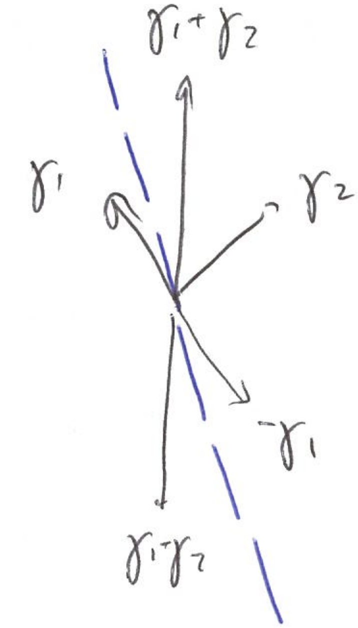
What if you just rotate the half plane to pass the ray $Z(\gamma_1)$?

$-Z(\gamma_1)$ becomes a member of cone of particles. So $-Z(\gamma_1)$ becomes basis element. What about other things on interior of cone?

$\rightsquigarrow$ Quiver mutation. If you pass $Z(\gamma_1)$, replace γ_1 by $-\gamma_1$ in spectrum.

Then replace $\gamma_i \longmapsto \begin{cases} \gamma_i + \langle \gamma_1, \gamma_i \rangle \gamma_1 & \text{if } \langle \gamma_1, \gamma_i \rangle \geq 0 \\ \gamma_i & \text{otherwise.} \end{cases}$

Remark This will work for quivers w/ potential as well. (When quiver has cycles, kill off relations to constrain representations.)



Ex $\underset{\gamma_1}{\circ}$ ——— $\underset{\gamma_2}{\circ}$

$\Big\}$ mutations

$\underset{-\gamma_1}{\circ} \qquad \underset{\gamma_2+\gamma_1}{\circ}$

$\Big\}$ compute arrows

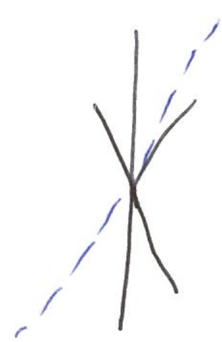
$\underset{-\gamma_1}{\circ} \longleftarrow \underset{\gamma_2+\gamma_1}{\circ}$

Every stable object eventually appears as a node of a quiver!

Knowing quiver mutations preserve stable objects, we know what stable objects are. Mutate again.

only true for some quivers and some geometries.

$\underset{\gamma_2}{\circ} \longrightarrow \underset{-\gamma_1-\gamma_2}{\circ}$



Once more:

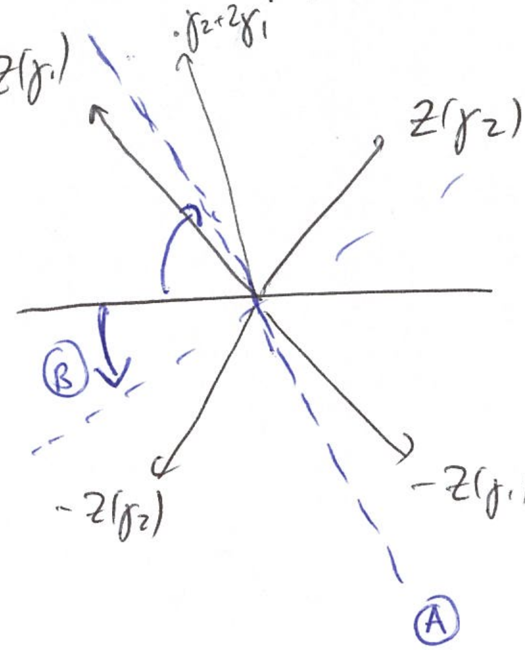
$\underset{-\gamma_2}{\circ} \longleftarrow \underset{-\gamma_1}{\circ}$ ← 180° rotation, just give antiparticles

Ex $\quad \overset{\gamma_1}{\circ} \rightrightarrows \overset{\gamma_2}{\circ}$

(on one side, $\mathrm{Coh}\,\mathbb{P}^1$
other side, special Lagrangians)

$\rightsquigarrow$

(A) $\quad \underset{-\gamma_1}{\circ} \leftleftarrows \underset{\gamma_2+2\gamma_1}{\circ}$



$\rightsquigarrow$

$\vdots$

will generate ∞ series of nodes,

$(n+1)\gamma_1 + n\gamma_2, \ \forall n \in \mathbb{N}.$

This accumulates, as $n \to \infty$, to the ray given by

$\gamma_1 + \gamma_2$.

[But we never get here by finitely many mutations.

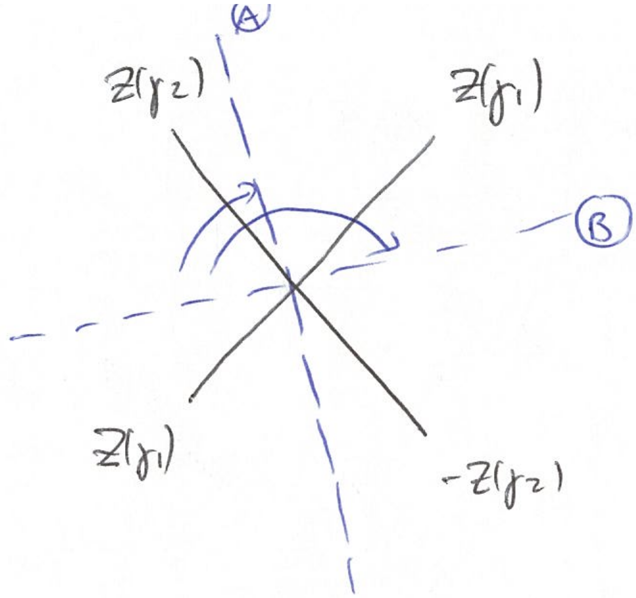
(B) (rotations other way) do the same:

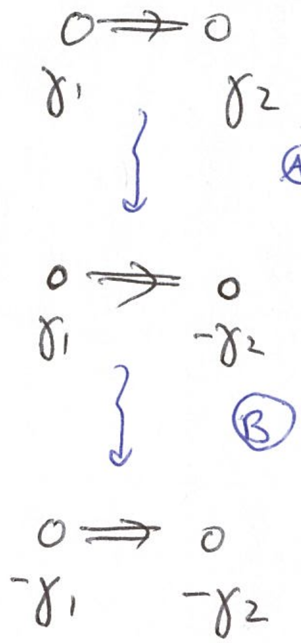
$n\gamma_1 + (n+1)\gamma_2$, also accumulating to $\gamma_1+\gamma_2$ ray.

So study rep theory of $\gamma_1+\gamma_2 \rightsquigarrow$ we get $\mathbb{P}^1$, as moduli space.

These are all the stable objects.

For





This theory is $N=2$, 4d $SU(2)$ gauge theory.

ie, Seiberg-Witten.

some config of fields are reps of $SU(2)$, and Lag. respects reps.

Moduli space, u local coordinate.

parametrizing different vacua of theory.

What is an exact function $\tau(u)$ where $\operatorname{Im}\tau(u)$ is gauge coupling?

(appears before $F \wedge * F$, i.e., $\operatorname{Im}\tau(u) F^2$ is term in

For particular values, only see Cartan algebra. Symmetry broken at low energies,

$SU(2) \rightsquigarrow U(1)$.

Labels of $U(1) \leftrightarrow$ 1-dim lattice electric charge

There's also weird field config of magnetic charge even though you never put it in by hand. Mind-blowing.

τ needs to be holomorphic. But cannot have τ holomorphic and non-positive curvature

└ since it's coeff of F^2, must be positive

At singularities, some BPS states become massless – called monopoles

What if you loop around u-plane? The monodromy.

$u \rightsquigarrow \mathbb{C}$ structure deformation

τ = period matrix of particular curve.

Physics: τ was section $\begin{pmatrix} a \\ a_D \end{pmatrix}$ of some bundle. A, B cycles

$$\begin{pmatrix} a \\ a_D \end{pmatrix} \sim \begin{pmatrix} \int_A \lambda_{SW} \\ \int_B \lambda_{SW} \end{pmatrix}$$

and central charge is

$$Z = \underbrace{e \cdot a}_{\text{electr}} + m a_D .$$

Eventually, Moore-Segal could see curves as inside some CY3, cycles would be special Lagrangians. So somehow BPS states have to do w/ things that become massless, and the only way we see remnants of this is via $Z(\gamma_1+\gamma_2)$