Bridgeland stability

1 9/17/2013 – BPS quivers, stability & wall crossing (cont.) – guest lecture by Murad Alim

1.1 Introduction

1.1.1 Ingredients

We recall what we did last time. The goal of the program is to characterize field theories – not in terms of minimizing an energy functional, but by using the objects (certain particles). This is in some sense analogous to the homological approach on the mathematical side (thinking about homology, etc.). The analogous question is: can you distinguish different categories by distinguishing their "bases" (i.e. plural of "basis")? The answer, in our context, will turn out to be *yes*.

Despite starting with Calabi–Yau geometries, we end up in 4 dimensions, with $N = 2$.

We have:

- The central ingredient is an r-dimensional moduli space $\mathcal{M}$, a deformation space of the underlying geometry. (Recall Hiro talked about sheaves; there the moduli space would be the deformation space of a symplectic structure. Or if we're talking about curves, this'd be the moduli of them.) This can also be interpreted as the Cartan algebra of rank r, coming from the gauge group $G \to U(1)^r$.
- We also have a lattice Γ of rank $2r$ of *electric* and *magnetic* charges, and the element $\gamma = (e^\alpha, m_a) \in \Gamma$. We can think of the electric/magnetic splitting as a choice of symplectic pairing $\langle \bullet, \bullet \rangle : \Gamma \times \Gamma \to \mathbb{Z}$.
- For each point $u \in \mathcal{M}$, we have the *central charge* $Z_u : \Gamma \to \mathbb{C}$. This has $M_u(\gamma) \geq |Z_u(\gamma)|$.

Remark 1. Hiro points out that this lattice is always given to us in the Kontsevich–Soibelman setup that we'll see sometime later. This runs as

$$\begin{array}{ccc} & \Gamma & \\ \swarrow & & \searrow \\ H^{even}(X, \mathbb{Q}) & & H_3(X^\vee, \mathbb{Z}). \end{array}$$

1.2 BPS quivers

By *spectrum* one should think of the set of all stable objects with respect to some heart. (One really ought to make a pun about having a "change of heart" here.) The "BPS states" are the lowest-energy particles, which should be thought of as generating the heart (and hence the category).

Suppose γ is in the spectrum. Then so is $-\gamma$. Since Z is linear (where we fix $Z = Z_u$ for some fixed $u \in \mathcal{M}$), we always have $Z(-\gamma) = -Z(\gamma) \in \mathbb{C}$. The crucial thing is that a different choice gives us a different choice of what we call *particles* versus *antiparticles*.

Let's choose a basis $\{\gamma_1, \ldots, \gamma_{2r}\}$, such that for all particles γ, we have $\gamma = \sum_{i=1}^{2r} n_i \cdot \gamma_i$ for some $n_i \in \mathbb{N}$.

Now, we define the following quiver: the nodes are labeled by the γ_i, and we have arrows $A^a_{ij} = \langle \gamma_i, \gamma_j \rangle$. Examples include $\bullet \to \bullet$ and $\bullet \Rightarrow \bullet$. The former we don't know how to characterize; the latter corresponds to Seiberg–Witten theory.

The obvious question is: What is this all useful for? Why isn't any random positive-integral linear combination going to give a BPS particle? And one we have this basis, how can we tell which $\gamma = \sum_{i=1}^{2r} n_i \cdot \gamma_i$ going to give a particle?

1.3 Quiver representation theory

To each γ, we can assign the following representation of our quiver: we assign $\mathbb{C}^{n_i}$ to the node γ_i, and we assign to-be-determined linear maps $B^a_{ij} : \mathbb{C}^{n_i} \to \mathbb{C}^{n_j}$. In fact, we will be interested in the *moduli space* of possible $\{B^a_{ij}\}_{i,j}$. Let us denote this representation as γ_R.

Note that if we have some $\gamma_S = \sum_{i=1}^{2r} m_i \cdot \gamma_i$ with $m_i \leq n_i$ (and not all equalities, so $\gamma_S \neq \gamma_R$), then we get a subrepresentation. These satisfy

$$\begin{array}{ccc} \mathbb{C}^{n_i} & \xrightarrow{B^a_{ij}} & \mathbb{C}^{n_j} \\ \uparrow & & \uparrow \\ \mathbb{C}^{m_i} & \xrightarrow[b^a_{ij}]{} & \mathbb{C}^{m_j}. \end{array}$$

We have the following notion of *stability* (due to Douglas, Fiol, and Romelsberger in 2000). We say that a representation γ_R is *stable* if for every subrepresentation γ_S, $\arg Z(\gamma_S) < \arg Z(\gamma_R)$. (Note that given a representation, we can forget the data of the linear maps and obtain an element of Γ; this gives us a notion of the central charge of a representation.)

And now, we can say what we demand of our B^a_{ij}. We define our moduli space to be

$$\mathcal{M} = \{\{B_{ij} : \mathbb{C}^{n_i} \to \mathbb{C}^{n_j}\}_{i,j} : \gamma_R \text{ is a stable representation}\} / \prod_{i=1}^{2r} GL(n_i, \mathbb{C}).$$

Example 1. Let us take the "A_2 quiver" $\bullet \to \bullet$, which gives rise to *Argyres–Douglas theory*. Write γ_1 for the source and γ_2 for the target. Then, take for example $\gamma = \gamma_1 + \gamma_2$. This is asking us for a single map $B : \mathbb{C} \to \mathbb{C}$, i.e. $B = M_1(\mathbb{C}) \cong \mathbb{C}$. To ask when this might be stable, let's imagine we've fixed a central charge Z, so e.g. for γ_R to be stable say $\arg Z(\gamma_1) < \arg Z(\gamma_1 + \gamma_2) < \arg Z(\gamma_2)$, so we have to consider γ_1. Then, stability requires that the diagram

$$\begin{array}{ccc} \mathbb{C} & \xrightarrow{B} & \mathbb{C} \\ \uparrow & & \uparrow \\ \mathbb{C} \xrightarrow[0]{} \mathbb{C} & & \end{array}$$

to commute, which of course forces $B = 0$.

On the other hand, if we had $\arg Z(\gamma_2) < \arg Z(\gamma_1 + \gamma_2) < \arg Z(\gamma_1$, then we'd see that the representation corresponding to γ_2 would require the commutativity of the diagram

$$\begin{array}{ccc} \mathbb{C} & \xrightarrow{B} & \mathbb{C} \\ \uparrow & & \uparrow \\ 0 & \longrightarrow & \mathbb{C}, \end{array}$$

and this can always be made to commute.

Thus, the moduli space is

$$\mathcal{M}_\gamma = \{\mathbb{C}^*\}/(GL_1(\mathbb{C}) \times GL_1(\mathbb{C})) = \text{pt}$$

(or perhaps really $BGL_1(\mathbb{C})$) when $\arg Z(\gamma_2) < \arg Z(\gamma_1)$, and the moduli space is

$$\mathcal{M}_\gamma = \emptyset$$

when $\arg Z(\gamma_1) < \arg Z(\gamma_2)$.

This is an example of *wall crossing*. This phenomenon will appear as soon as the ordering of the phases switches around.

1.4 Computing spectra

The example we looked at was relatively easy, but even the quiver $\bullet \Rightarrow \bullet$ becomes much more difficult; even for slightly larger quivers the computations become unmanageable. However, in certain situations there is a shortcut, which we explain now.

Recall that we chose half of the objects (those whose central charge lives in the upper half-plane (UHP)) more or less at random. In particular, since everything is in the $\mathbb{Z}$-span of the generators, these form some cone in which all the central charges live.

So but now, suppose we change the half-plane, so much that (let's say) $Z(\gamma_1)$ leaves the UHP. Then of course $Z(-\gamma_1)$ enters the UHP. But of course, this changes our decompositions of objects by basis elements. The key insight here is that this is reflected on the quiver side via what we'll call a *quiver mutation*. This is obtained by associating $\gamma_i \mapsto \gamma_i + \langle \gamma_1, \gamma_i \rangle \cdot \gamma_1$ whenever $\langle \gamma_1, \gamma_i \rangle \geq 0$, and keeping γ_i fixed otherwise.

Example 2. Let's go back to the quiver $\bullet \to \bullet$. Say $\arg Z(\gamma_2) < \arg Z(\gamma_1)$. If we rotate so that $Z(\gamma_1)$ just barely leaves the left side of the UHP, then we change our basis to $\{-\gamma_1, \gamma_2 + \gamma_1\}$. When we compute the arrows for this new basis, we get the quiver $\bullet \leftarrow \bullet$. (Of course, now that $\gamma_2 + \gamma_1$ is a basis element, we get for free that it's stable.) Continuing to rotate gives us the quiver $\gamma_2 \to (-\gamma_1 - \gamma_2)$, and then $(-\gamma_2) \leftarrow (-\gamma_1)$; by now we've rotated by $180°$.

Note that in the previous example, *every stable object eventually appeared as a node of the quiver*. This is only true for certain types of quivers and certain types of geometries, but when it works out it's extremely useful.

Example 3. Let's now turn to the quiver $\bullet \Rightarrow \bullet$. This one also has two chambers in the moduli space. Let's skip past the quiver representation theory and just run the quiver mutations.

Suppose we have $\arg Z(\gamma_2) < \arg Z(\gamma_1)$. Then, after $\gamma_1 \Rightarrow \gamma_2$ we get $(-\gamma_1) \Leftarrow (\gamma_2 + 2\gamma_1)$, etc. This ends up generating an infinite sequence of nodes, which will all have the form $(n+1)\cdot\gamma_1 + n\cdot\gamma_2$ for $n \in \mathbb{N}$. However, these accumulate towards just being $n \cdot (\gamma_1 + \gamma_2)$ in the limit $n \to \infty$. It turns out that even if we rotated in the opposite way, we'd still get the same ray on which they accumulate (instead via $n \cdot \gamma_1 + (n+1) \cdot \gamma_2$). It's not an accident that this is the "problem" object that we were considering earlier.

Studying the representation theory for the element $\gamma = \gamma_1 + \gamma_2$, one obtains the moduli space $\mathbb{P}^1$.

On the other hand, suppose $\arg Z(\gamma_1) < \arg Z(\gamma_2)$. Then we start with $\gamma_1 \Rightarrow \gamma_2$, and then get $\gamma_1 \Leftarrow (-\gamma_2)$, and then get $(-\gamma_1) \Rightarrow (-\gamma_2)$. In this case, things simply repeat instead.

This theory corresponds to a 4-dimensional, $N = 2$, $SU(2)$-gauged theory, called *Seiberg–Witten theory*. This comes from $SU(2) \to U(1)$. This corresponds to *symmetry-breaking*. Any charge in $U(1)$ is called *electric*. From the Lagrangian perspective, there's some weird stuff going on where we get magnetic charges "without having put them there". Here the moduli space is $U(1)$, and in this special case u becomes a local coordinate on the moduli space. In terms of the gauge theory, u is parametrizing different "vacua" of the theory.

What Seiberg–Witten began by asking is: What is an exact function $\tau(u)$ such that we have $\mathrm{Im}(\tau(u)) \cdot F \wedge \star F$ as our whatever-the-hell physicsy thing? It turns out that τ has to be holomorphic. It turns out that we can't have a τ which is both holomorphic and crosses into the UHP, so we must have singularities. The strike of genius was interpreting physically what these singularities should mean; the interpretation is that the BPS particles become massless, and we call these things *monopoles*.

2 9/19-2013 – Back to basics: abelian categories, quiver representation theory, and also maybe abelian categories admitting stability conditions

2.1 Abelian categories

Definition 1. Let $\mathcal{C}$ be a category enriched over abelian groups. We say that $\mathcal{C}$ is *additive* if:

(0) $\mathcal{C}$ has a zero object (i.e. an object which is both initial and terminal), and

(1) $\mathcal{C}$ has finite products.

Exercise 1. If $\mathcal{C}$ is additive, then:

(2) $\mathcal{C}$ has finite coproducts, and

(3) these coincide (up to natural isomorphism) with finite products.

Exercise 2. If a category $\mathcal{C}$ satisfies properties (0)-(3), then it is naturally enriched over abelian monoids. (One must simply demand that these are abelian groups; for instance, the category of abelian monoids itself satisfies properties (0)-(3) but is only enriched over itself.)

Definition 2. An additive category $\mathcal{C}$ is called *abelian* if:

(4) for any map $A \xrightarrow{f} B$, the kernel

$$\ker(f) := \lim(0 \to B \xleftarrow{f} A)$$

and the cokernel

$$\mathrm{coker}(f) := \mathrm{colim}(0 \leftarrow A \xrightarrow{f} B)$$

both exist, and

(5) the natural map

$$\mathrm{coim}(f) := \mathrm{coker}(\ker(f) \to A) \to \ker(B \to \mathrm{coker}(f)) =: \mathrm{im}(f)$$

(from the coimage to the image) is an isomorphism.

Exercise 3.

- The canonical map
$$\ker(f) \xrightarrow{k} A$$
is a monomorphism (i.e. $k \circ g = k \circ h \Rightarrow g = h$).
- The canonical map
$$B \xrightarrow{c} \operatorname{coker}(f)$$
is an epimorphism (i.e. $g \circ x = h \circ c \Rightarrow g = h$).

Example 4. Here is an example of property (5). By the universal property of $\operatorname{coker}(k)$, i.e. the fact that the diagram

$$\begin{array}{ccc} \ker(f) & \xrightarrow{k} & A \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & \operatorname{coker}(k) \end{array}$$

is a pushout, the map $A \xrightarrow{f} B$ induces a unique map $\operatorname{coker}(k) \to B$ making the triangle

$$\begin{array}{ccc} A & & \\ \downarrow & \searrow^{f} & \\ \operatorname{coker}(k) & \dashrightarrow & B \end{array}$$

commute. But $c \circ f = 0$, so $\operatorname{coker}(k) \to B \to \operatorname{coker}(f)$ is zero.

Now that we have notions of kernels and cokernels, we can make the following general definition.

Definition 3. A *short exact sequence* in $\mathcal{C}$ is a sequence of composable morphisms

$$0 \xrightarrow{f_0} A \xrightarrow{f_1} B \xrightarrow{f_2} C \xrightarrow{f_3} 0$$

such that $\ker(f_{i+1}) \cong \operatorname{coker}(f_i)$ for all i.

Definition 4. The *Grothendieck group* of $\mathcal{C}$, denoted $K_0(\mathcal{C})$ is the abelian group generated by the set of objects of $\mathcal{C}$, with the relations that the existence of a short exact sequence

$$0 \to A \to B \to C \to 0$$

implies that $[A] + [C] = [B]$.

Exercise 4.

1. Let k be a field. Then $K_0(\mathtt{Vect}_k) \cong 0$ and $K_0(\mathtt{Vect}_k^{f.d.}) \cong \mathbb{Z}$.
2. Let R be a P.I.D. Show that the same result holds.
3. Using the previous part, show that $\mathtt{Ab}^{f.g.}$ doesn't admit a stability condition.
4. Compare $K_0(\mathtt{Ab}^{fin})$ with $K_0(\mathtt{Ab}^{f.g.})$.

Proof. For (1a), we use Eilenberg's swindle: $V \oplus (\bigoplus_{\mathbb{N}} V) \cong \bigoplus_{\mathbb{N}} V$. Thus $[V] = 0 \in K_0(\texttt{Vect}_k)$.

For (1b), we define dim : ob $\texttt{Vect}_k^{f.d.} \to \mathbb{Z}$; since we can form any short exact sequence with finite-dimensional vector spaces of the appropriate dimensions (i.e. $0 \to k^{\oplus m} \to k^{\oplus(m+n)} \to k^{\oplus n} \to 0$) and all short exact sequences split, we obtain the result.

For (2), we use the same proof as for (1b), using the *rank* function: for k the fraction field of R and $M \in \texttt{Mod}_R$, we define rk $M = \dim_k(M \otimes_R k)$.

For (3), we see that $[\mathbb{Z}/n\mathbb{Z}] = 0 \in K_0(\texttt{Ab}^{f.g.})$, which contradicts the requirement that the central charge of all non-(isomorphic-to-)0 objects is nonzero in $\mathbb{C}$.

□

2.2 Quiver representation theory

Definition 5. A *quiver* Q is a 1-dimensional semisimplicial set, i.e. the data of:

- two sets Q_0 and Q_1, and
- two maps $s, t : Q_1 \to Q_0$ (standing for *source* and *target*).

We think of Q_0 as the set of vertices and Q_1 as the set of arrows; together these form a directed graph.

In this class, both Q_0 and Q_1 will always be *finite*.

Let's fix a base field k once and for all. (This will almost always be over $\mathbb{C}$, or at least some field of characteristic zero.)

Definition 6. A *representation* of a quiver Q is an assignment of a vector space V_i to each element $i \in Q_0$ and a linear map $f_\alpha : V_{s(\alpha)} \to V_{t(\alpha)}$ for every morphism $\alpha \in Q_1$.

Definition 7. Given two representations $(\{V_i\}, \{f_\alpha\})$ and $(\{V_i'\}, f_\alpha'\})$, a *morphism* from the first to the second is a collection of maps $g_i : V_i \to V_i'$ for each $i \in Q_0$ such that for all $\alpha \in Q_1$, the diagram

$$\begin{array}{ccc} V_i & \xrightarrow{f_\alpha} & V_j \\ \Big\downarrow{\scriptstyle g_i} & & \Big\downarrow{\scriptstyle g_j} \\ V_i' & \xrightarrow[f_\alpha']{} & V_j' \end{array}$$

commutes.

Example 5. Let Q be the quiver with one vertex and one arrow. Then a representation of Q is precisely a choice of a vector space $V \in \texttt{Vect}$ and an endomorphism $f \in \text{End}(V)$.

Definition 8. A *path* in Q is an element

$$(\alpha_n, \dots, \alpha_1) \in Q_1 \times_{Q_0} Q_1 \times_{Q_0} \cdots \times_{Q_0} Q_1$$

of the n-fold fiber product over the source map on one side and the target map on the other; we take the convention that $t(\alpha_i) = s(\alpha_{i+1})$. Thus we should think of this as

$$\bullet \xrightarrow{\alpha_1} \cdots \xrightarrow{\alpha_n} \bullet.$$

We call the number n the *length* of the path. By convention, a path of length 0 is a choice of vertex. (This follows from the definition, since the fiber product of zero copies of Q_1 over Q_0 is just Q_0.) We'll write the empty path at vertex i as e_i.

Definition 9. Let $(kQ)_l$ be the k-vector space spanned by the paths of length l. These assemble into the *path algebra*, denoted kQ, which is the free associative algebra generated by $Q_0 \cup Q_1$, subject to the relation that "multiplication is concatenation of paths":

- $e_i^2 = e_i$ for all $i \in Q_0$;
- $e_i e_j = 0$ if $i \neq j$;
- $\alpha e_i = \alpha$ if $s(\alpha) = i$;
- $e_j \alpha = \alpha$ if $t(\alpha) = j$;
- $\beta\alpha = 0$ if $s(\beta) \neq t(\alpha)$.

(Our notation is that of composition of functions: the path on the right is the one that happens first.)

Remark 2. kQ is unital, with identity element $\sum_{i\in Q_0} e_i$. (Note that this assumes Q_0 is finite!) Note that we'd have a distinct unit element if we had taken the free unital associative algebra in the definition.

Remark 3. As a vector space, $kQ \cong \bigoplus_{l\geq 0}(kQ)_l$.

Note that there is a natural equivalence of categories $\mathtt{Rep}(Q) \simeq kQ\text{-}\mathtt{mod}$ (left modules, by our conventions).

Proposition 1. *If Q is acyclic (i.e. has no oriented cycles), then* $\mathtt{Rep}^{f.d.}(Q) \simeq kQ\text{-}\mathtt{mod}^{f.g.}$.

Exercise 5. Why does this fail if Q has oriented cycles?

Proof of proposition. Say $(\{V_i\}, \{f_\alpha\})$ is a finite-dimensional Q-representation. Let $M = \bigoplus_{i\in Q_0} V_i$, with a left action of kQ determined by the following:

$$e_i \cdot m = \begin{cases} m & m \in V_i \\ 0 & m \in V_j \text{ for } j \neq i \end{cases}$$

and

$$\alpha \cdot m = \begin{cases} f_\alpha m & m \in V_{s(\alpha)} \\ 0 & m \in V_j \text{ for } j \neq s(\alpha). \end{cases}$$

(Of course, we extend this to all of M by linearity.)

Conversely, given a left kQ-module M, we set $V_i = e_i \cdot M$, and for all paths α of length 1, we set $f_\alpha m = \alpha \cdot m$ for all $m \in V_{s(\alpha)}$. □

Definition 10. The *fundamental quiver representation* of a quiver Q is equivalent to kQ considered as a left kQ-module.

Exercise 6. For the following quivers Q, determine the fundamental quiver representation.

1. The quiver with two vertices and a single arrow between them.
2. The quiver with two vertices and two parallel arrows between them.
3. The quiver with one vertex and one arrow.
4. The quiver with two vertices and two arrows between them going in opposite directions.

Next time, we'll show that any quiver (acyclic or not) has an abelian category of representations which admits Bridgeland stability conditions.