BRIDGELAND STABILITY
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LECTURE 6

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After having rushed through many exciting connections with physics and having made several unjustified assertions, we will now slow down and be more rigorous.

1. Categorical Background

We start by recalling several basic definitions from category theory - we will ignore set-theoretic issues.

Definition 1.

- An object 0 in a category $\mathcal{C}$ is called *zero object* if there is a unique morphism from and to any other object in $\mathcal{C}$.
- A *product* of two objects $X_1, X_2 \in \operatorname{ob}\mathcal{C}$ is a diagram

  $$X_1 \xleftarrow{\pi_1} X_1 \times X_2 \xrightarrow{\pi_2} X_2$$

  such that for any other diagram

  $$X_1 \xleftarrow{f_1} Y \xrightarrow{f_2} X_2$$

  there is a unique $f : Y \to X_1 \times X_2$ with $\pi_i \circ f = f_i$ for $i = 1, 2$. Like all limits, products are, if they exist, unique up to unique isomorphism.
- *Coproducts* are defined dually.

One defines *pullbacks* and *pushouts* along similar lines. If a category has a zero object, it is said to be *pointed* and we can give diagrammatic definitions of several well-known concepts from algebra:

Definition 2. Let $\mathcal{C}$ be a pointed category and $f : A \to B$ a morphism in $\mathcal{C}$.
The *kernel of* f is, if it exists, given by the following pullback diagram:

$$\begin{array}{ccc} \operatorname{Ker}(f) & \xrightarrow{\ker(f)} & A \\ \downarrow & \lrcorner & \downarrow f \\ 0 & \longrightarrow & B \end{array}$$

The *cokernel of* f is, if it exists, the following pushout diagram:

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \downarrow & & \downarrow \operatorname{coker}(f) \\ 0 & \ulcorner\!\longrightarrow & \operatorname{Coker}(f) \end{array}$$

Definition 3. We say that a category $\mathcal{C}$ is *enriched over abelian groups* if all its Hom-sets $hom_{\mathcal{C}}(X,Y)$ are endowed with the structure of abelian groups in a way which make the composition maps

$$hom_{\mathcal{C}}(X,Y) \times hom_{\mathcal{C}}(Y,Z) \to hom_{\mathcal{C}}(X,Z)$$

bilinear.

Definition 4. Such a category $\mathcal{C}$ which is enriched over abelian groups is said to be *additive* if it has a zero object and finite products.

Exercise 5. *Show that if the category $\mathcal{C}$ is additive, then:*

(1) it has finite coproducts
(2) its binary products and coproducts are naturally isomorphic - they are therefore called biproducts

We finally arrive at the following central notion:

Definition 6. An additive category $\mathcal{C}$ is called *abelian* if

- it has kernels and cokernels.
- for every morphism f, the canonical arrow
$$\mathrm{Coker}\left(\mathrm{Ker}(f) \xrightarrow{\ker(f)} A\right) \to \mathrm{Ker}\left(B \xrightarrow{\mathrm{coker}(f)} \mathrm{Coker}(f)\right)$$
is an isomorphism.

The last condition ensures that we have a "well-behaved" notion of an image:

Definition 7. We define the *image* of a map f in an abelian category as

$$\mathrm{im}(f) = \mathrm{Ker}(\mathrm{coker}(f))$$

Exercise 8. *For any morphism f:*

- *the map* $\mathrm{Ker}(f) \xrightarrow{\ker(f)} A$ *is a monomorphism (i.e. has left cancellation)*
- *the map* $B \xrightarrow{\mathrm{coker}(f)} \mathrm{Coker}(f)$ *is an epimorphism (i.e. has right cancellation).*

Let us now construct the canonical map used in the definition of an abelian category explicitely. We have a pullback diagram

$$\begin{array}{ccc} \mathrm{Ker}(f) & \xrightarrow{\ker(f)} & A \\ \downarrow & \lrcorner & \downarrow f \\ 0 & \longrightarrow & B \end{array}$$

and a pushout square

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \downarrow & & \downarrow \mathrm{coker}(f) \\ 0 & \ulcorner\!\longrightarrow & \mathrm{Coker}(f) \end{array}$$

By the universal property of coker(ker(f)), we obtain a unique map:

$$\begin{array}{ccccc} \mathrm{Ker}(f) & \xrightarrow{\ker(f)} & A & & \\ \downarrow & & \downarrow \mathrm{coker}(\ker(f)) & \searrow^{f} & \\ 0 & \longrightarrow & \mathrm{Coker}(\ker(f)) & \xrightarrow{!\exists} & B \end{array}$$

It is clear that

$$\mathrm{Coker}(\ker(f)) \to B \to \mathrm{Coker}(f)$$

is zero. By the universal property of Ker(coker(f)), this gives a unique map

$$\mathrm{Coker}(\ker(f)) \longrightarrow \mathrm{Ker}(\mathrm{coker}(f))$$

2. The Grothendieck group

We fix an abelian category $\mathcal{C}$.

Definition 9. A *short exact sequence* is a sequence of maps

$$0 \xrightarrow{f_0} A \xrightarrow{f_1} B \xrightarrow{f_2} C \xrightarrow{f_3} 0$$

such that $\ker(f_{i+1}) \cong \mathrm{im}(f_i)$ for all i.

Definition 10. The *Grothendieck group* of $\mathcal{C}$ is the abelian group $K_0(\mathcal{C})$ generated by $\mathrm{ob}\,\mathcal{C}$ modulo the relation

$$[A] + [C] = [B]$$

whenever there is some short exact sequences $0 \to A \to B \to C \to 0$.

We have the following elementary results:

Proposition 11.

(1) For any field k, we have:
   a) $K_0(Vect_k) \cong 0$
   b) $K_0(Vect_k^{f.d.}) \cong \mathbb{Z}$

(2) For any principal ideal domain R we have $K_0(Mod_R^{f.g.}) \cong \mathbb{Z}$

(3) The category $Ab^{f.g.}$ does not admit a Bridgeland stability condition.

Proof.
(1) a): This is known as the Eilenberg swindle trick. For any vector space V, we can chose an isomorphism

$$V \oplus \left(\bigoplus_{n \geq 1} V \right) \cong \bigoplus_{n \geq 1} V$$

which implies that $[V] = 0$ in the Grothendieck group $K_0(Vect_k)$.
(1) b): The (virtual) dimension function

$$\dim : \mathbb{Z}\,\mathrm{ob}\,\mathcal{C} \to \mathbb{Z}$$

$$V \mapsto \dim V$$

out of the free abelian groups spanned by the objects of $\mathcal{C}$ is surjective and its kernel is spanned by all combinations $[A] + [C] - [B]$ for which there is a short exact sequence $0 \to A \to B \to C \to 0$.

(2): This is proven in the same way as (1) b) by using rank instead of dimension.
(3): The short exact sequence

$$0 \to \mathbb{Z} \xrightarrow{\cdot n} \mathbb{Z} \to \mathbb{Z}/n\mathbb{Z} \to 0$$

shows that $[\mathbb{Z}/n\mathbb{Z}] = 0 \in K_0$ which implies $Z(\mathbb{Z}/n\mathbb{Z}) = 0$ for any stability condition. This contradicts the axioms. □

Exercise 12. *Compute the Grothendieck group $K_0(Ab^{fin})$.*

3. Quivers

Definition 13. A *quiver* Q is just a directed multigraph. More formally, it is determined by a pair of sets (Q_0, Q_1) (the vertices and the arrows) and a pair of maps $s, t : Q_1 \to Q_0$ (the source and the target map).

In this class, the sets Q_0, Q_1 will usually be assumed to be finite.

Example 14. Take vertices $Q_0 = \{1, 2\}$ and edges $Q_1 = \{e, f, g, h\}$ with a source function s given as:

$$e \mapsto 1 \qquad f \mapsto 1$$
$$g \mapsto 1 \qquad h \mapsto 2$$

and a target function t defined by:

$$e \mapsto 1 \qquad f \mapsto 2$$
$$g \mapsto 2 \qquad h \mapsto 2$$

Definition 15. A *representation* of a quiver Q is consists of the data of

- vector spaces V_i for all $i \in Q_0$
- linear maps $V_{s(\alpha)} \to V_{t(\alpha)}$ for all $\alpha \in Q_1$.

Definition 16. A *morphism* between two representations $(\{V_i\}, \{f_\alpha\})$ and $(\{W_i\}, \{g_\alpha\})$ of a quiver Q is a collection of linear maps $\{\phi_i : V_i \to W_i\}$ such that

$$\begin{array}{ccc} V_i & \xrightarrow{f_\alpha} & V_j \\ {\scriptstyle \phi_i}\downarrow & & \downarrow{\scriptstyle \phi_j} \\ W_i & \xrightarrow{g_\alpha} & W_j \end{array}$$

commutes for all α with $s(\alpha) = i, t(\alpha) = j$.

Definition 17. A *path of length* n in a quiver Q is an element

$$(\alpha_n, ..., \alpha_1) \in Q_1 \times_{Q_0} .... \times_{Q_0} Q_1$$

Note that paths of length 0 correspond to vertices, and we will write e_i for the path corresponding to the vertex $i \in Q_0$.
Let $(kQ)_l$ be the vector space spanned by all paths of length l in a quiver Q.

Definition 18. The *path algebra* kQ is the free nonunital associative algebra generated by $Q_0 \cup Q_1$ such that:

- $e_i^2 = e_i$ for all $i \in Q_0$ and $e_i e_j = 0$ for $i \neq j$.
- $\alpha e_i = \alpha$ if $s(\alpha) = i$ and $e_j \alpha = e_j$ if $t(\alpha) = j$
- $\beta\alpha = 0$ if $s(\beta) \neq t(\alpha)$

Remark 19. In the case where Q_0 finite, the algebra $kQ = \bigoplus_{l \geq 0} (kQ)_l$ is actually unital with unit $1 = \sum_{i \in Q_0} e_i$.

Proposition 20. *There is an equivalence of categories*

$$Rep(Q) \cong LMod_{kQ}$$

between the representations of a quiver and the left modules over its path-algebra.

Proof. If $(\{V_i\}, \{f_\alpha\})$ is a representation of W, we define a left module

$$M = \bigoplus_{q \in Q_0} V_i$$

whose kQ-action is given by

$$e_i m = \begin{cases} m & \text{if } m \in V_i \\ 0 & \text{else} \end{cases} \qquad \alpha m = \begin{cases} f_\alpha(m) & \text{if } m \in V_{s(\alpha)} \\ 0 & \text{else} \end{cases}$$

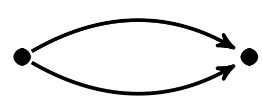
Conversely given a left module M over the path-algebra of Q, set $V_i = e_i \cdot M$. If α is an edge of our quiver, it is also a path of length 1. This allows us to define $f_\alpha(m) = \alpha \cdot m \in V_{t(\alpha)}$ for all $m \in V_{s(\alpha)}$. □

Exercise 21. *If Q is acyclic, then $Rep^{f.d.}(Q) \cong LMod_{kQ}^{f.g.}$.*

Exercise 22. *Describe the quiver representation corresponding to kQ, considered as a $kQ-$module, in each of the following cases:*

(1)

(2)



(3)

(4)