BRIDGELAND STABILITY CONDITIONS
LECTURE 7

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1. Quivers and path algebras

Let us recall what we did lat time.

Definition 1.1. A *quiver* Q is

$$Q = (Q_0, Q_1, s, t : Q_1 \to Q_0)$$

where

(1) Q_0 is the set of vertices,
(2) Q_1 is the set of edges,
(3) $s, t : Q_1 \to Q_0$ sends an edge t to its source $s(e)$ and target $t(e)$.

Definition 1.2. To a quiver Q, we define the *path algebra* kQ as the free (not necessarily unital) associative algebra generated by paths of lengths 0 and 1, modulo concatenation of paths.

Let $Rep^{fd}(Q)$ be the category of finite dimensional representations of Q, and kQ^{fd}-mod is the category of finite dimensional modules over kQ.

Proposition 1.3. *There is an equivalence of categories between $Rep^{fd}(Q)$ and kQ^{fd}-mod.*

Proof. Given a module M, we produce a quiver representation by letting $V_i = e_i M$, where e_i is the path of length zero at the vertex i. Note that

$$M = \oplus_{i \in Q_0} V_i$$

The action of the length one paths produces linear maps $f_e : V_{s(e)} \to V_{t(e)}$ for each edge e. The data V_i, f_e is representation of Q.

Conversely, given a Q-rep (V_i, f_α), we let

$$M = \oplus_{i \in Q_0} V_i$$

and define the action by length one paths correspond to the linear maps f_α. □

To each quiver Q, there is a canonical representation of the path algebra: the regular one. That is, view kQ as a kQ-module. It is fun to describe it in terms of quiver representations.

Exercise 1. Describe the regular representation of the following quivers:

(a) $v_0 \bullet \longrightarrow \bullet v_1$
(b) $v_0 \bullet \rightrightarrows \bullet v_1$

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(c) $v_0 \bullet \rightleftarrows \bullet v_1$

(d) $v_0 \bullet \circlearrowright$

Solution 1. (a) $ke_0 \bullet \longrightarrow \bullet ke_1 \oplus k\alpha$ where the map sends e_0 to α.

(b) $ke_0 \bullet \rightrightarrows \bullet ke_1 \oplus k\alpha \oplus k\beta$ and the top map sends e_0 to α, and the bottom sends e_0 to β.

(c) $V_0 \bullet \rightleftarrows \bullet V_1$ where

$$V_0 = ke_0 \oplus k(\beta\alpha) \oplus k(\beta\alpha\beta\alpha) \oplus \dots$$
$$V_1 = ke_1 \oplus k\alpha \oplus k\alpha\beta\alpha \oplus \dots$$

and the top map is "multiplication by α", and the bottom is "multiplicaiton by β".

(d) $V_0 \bullet \circlearrowright$ where $V_0 = k[\alpha]$ and the map is multiplication by α.

These examples suggest that if the quiver has a cycle, then the regular representation is infinite dimensional. We will show this later.

Remark 1.4. One can ask if the regular representation is nevertheless a colimit of finite dimensional representations (even if the quiver has cycles). We haven't seen a counterexample, but the translation to modules indicates that the answer is expected to be no. In general modules are colimits of finitely generated modules, but not of finite modules.

2. Simple objects, Jordan-Holder filtrations and finite length categories

Definition 2.1. $\mathcal{C}$ abelian category. An object E is simple if any monomorphism $i : E' \to E$ is either the inclusion of the zero object, or an isomorphism.

Example 2.2. For any Q, let $S(i)$ be the representation w/ k at the vertex $i \in Q_0$. It is simple! If the quiver is acyclic, these are all the simple ones.

Example 2.3. For $Q = \bullet \rightleftarrows \bullet$, the cycle with identities is simple.

Proposition 2.4. *Let V be a simple representation of a quiver Q. Assume Q_0 is finite, or V is finite dimensional. If Q has no cycles, then every simple representation is $S(i)$ for some vertex $i \in Q_0$.*

Proof. If Q has no cycles, Q_0 is a poset (set $i \geq j$ if there is a path from i to j). Given and representation V, let i be maximal such that $V_i \neq 0$. Then $S(i)$ is a subrepresentation. □

One of the reasons to care about simple objects is that they generate the Grothedieck group. More on that later.

Definition 2.5. Let $\mathcal{C}$ be an abelian category, and $E \in ob\mathcal{C}$. A Jordan-Holder filtration for E is a finite sequence

$$0 = E_0 \subset E_1 \subset \dots \subset E_n = E$$

such that E_{i+1}/E_i is simple.

Remark 2.6. Unlike Harder-Narasimhan filtration, J-H filtration is not unique. But the simple factors

$$E_{i+1}/E_i$$

are unique up to re-ordering.

On the other hand, in JH-filtration the quotients are simple, which implies stable, while in HN-filtration the quotients are only semistable.

Definition 2.7. An abelian category $\mathcal{C}$ is called *finite length* if every object admits a J-H filtration.

Proposition 2.8. *Given any associative algebra A, the category A^{fd}-mod of finite dimensional A modules is finite length.*

Proof. Given E, it is simple or not. If it is simple, we are done. Otherwise, choose a simple sub object $S_0 = E_0$. This exists because if every submodule had a non-zero submodule, we would get an infinite descending sequence, which contradicts finite length.

Let

$$0 \to S_0 \to E \to E/S_0 \to 0$$

By induction on the length of E/S_0, we can assume that it has a J-H filtration:

$$0 \subset \tilde{S}_1 \subset \ldots \subset \tilde{S}_n \subset E/S_0$$

We now pull this filtration back to E. That is, let

$$S_i = \tilde{S}_i \otimes_{E/S_0} E$$

We get the desired JH-filtration.

$$0 \subset S_0 \subset S_1 \subset \ldots \subset S_n \subset E$$

□

In particular $Rep^{fd}Q$ is finite length.

If $\mathcal{C}$ is finite-length, then $K_0(\mathcal{C})$ is spanned by simple objects, essentially because we can use the filtration to build E_{i+1} as an extension of E_i by E_{i+1}/E_i.

Proposition 2.9. *If $\mathcal{C}$ is finite length, then*

$$K_0(\mathcal{C}) = \mathbb{Z}^{simp(\mathcal{C})}$$

where $simp(\mathcal{C})$ is set of isomorphisms classes of simple objects.

Proof. Consider the map

$$\mathbb{Z}^{simp(\mathcal{C})} \to K_0(\mathcal{C})$$

sending an element $\mathbb{Z}^{simp(\mathcal{C})}$ to the corresponding linear combination the classes of the simple objects.

The inverse map sends $[E]$ to the sum of the factors in its J-H filtration. □

3. Constructing Bridgeland stability conditions

Recall that a Bridgeland stability condition on an abelian category $\mathcal{C}$ is a map

$$Z : ob\mathcal{C} \to \mathbb{C}$$

such that

(0) $image(Z) \subset \bar{\mathbb{H}} - \mathbb{R}_{>0}$
(1) $Z(E) = 0 \implies E = 0$
(2) Z descends to a map $Z : K_0(\mathcal{C}) \to \mathbb{C}$ of abelian groups.
(3) Every E has a H-N filtration, i.e.,

$$0 = E_0 \subset E_1 \subset \ldots \subset E_n = E$$

such that
- E_{i+1}/E_i is Z-semistable
- $\varphi(E_{i+1}/E_i) > \varphi(E_{i+2}/E_{i+1})$, where $\varphi = \arg Z$ is the phase map.

Suppose now we are given an abelian category $\mathcal{C}$ and a map $Z : ob\mathcal{C} \to \mathbb{C}$. We want to find easy to check conditions to tell if $(\mathcal{C}, Z)$ is a stability condition.

Definition 3.1. For $E \in ob\mathcal{C}$, let $\mathcal{H}_Z(E)$ be the convex hull of $\{Z(A) | A \subset E\}$, (the zero object included).

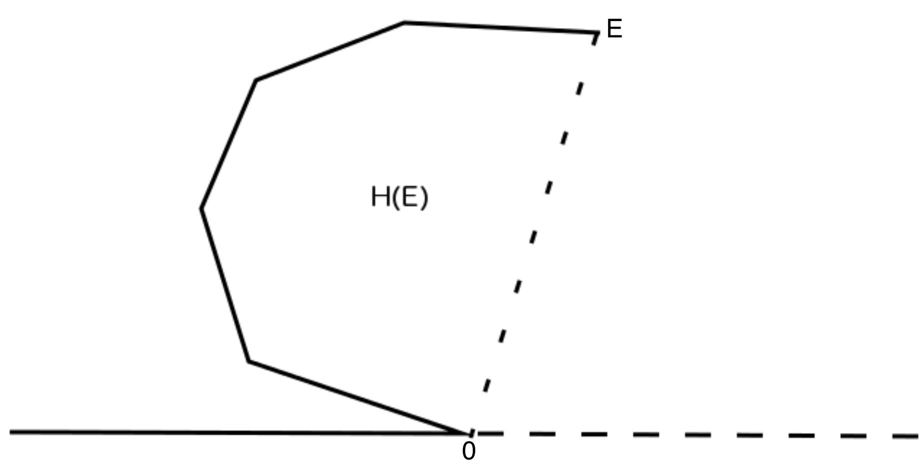

Figure 1. $\mathcal{H}(E) \cap \{re^{i\pi\theta}\}$

Theorem 3.2. *Let $\mathcal{C}$ be abelian, suppose we have Z satisfying $(0), (1)$ and (2). And*

($\star$) For any $E \in ob\mathcal{C}$, the region

$$\mathcal{H}(E) \cap \{re^{i\pi\theta} | \theta \in [\varphi(E), 1]\}$$

is compact, and the boundary is a finite polygon.

Then Z satisfies (3).

Proof. Fix E. Consider the set of extremal points of $\mathcal{H}(E) \cap \{re^{i\pi\theta} | \theta \in [\varphi(E), 1]\}$, and order them as $v_1, v_2, \ldots, v_{n-1}$ such that

$$arg(v_i) > arg(v_{i+1})$$

We let $v_0 = 0$, and $v_n = Z(E)$. As in the following diagram:

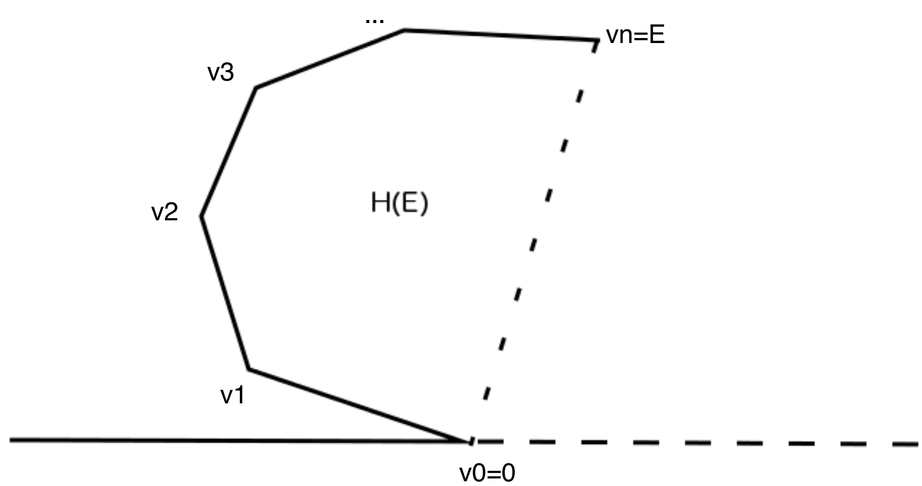

FIGURE 2. Numbering of vertices

Lemma 3.3. *Fix $A_i \subset E$ such that $Z(A_i) = v_i$. Then*
(a) $A_i \subset A_{i+1}$
(b) $arg(A_{i+2}/A_{i+1}) < arg(A_{i+1}/A_i)$
(c) A_{i+1}/A_i *is semistable.*

This will be our H-N filtration, and complete he proof of Theorem 3.2. □

Proof of Lemma 3.3. Consider $B_i = A_i \cap A_{i+1}$, and $C_i = A_i + A_{i+1}$ (that is, the pullback over E, and the image of the direct sum). To prove part a, we will show $B_i = A_i$ (and $C_i = A_{i+1}$).

Looking at the sequences:

$$0 \to B_i \to A_i \to A_i/B_i \to 0 \tag{1}$$

$$0 \to A_{i+1} \to C_i \to C_i/A_{i+1} \to 0 \tag{2}$$

We get

$$im(Z(B_i)) \leq im(Z(A_i)) \tag{3}$$

$$im(Z(C_i)) \geq im(Z(A_{i+1})) \tag{4}$$

and equalities holds only if $Z(B_i) = Z(A_i)$ and $Z(C_i) = Z(A_{i+1})$. Looking back at the sequences (1), we see that these equalities in central charge imply $B_i = A_i$ and $C_i = A_{i+1}$.

Graphically, this limits the region in the complex plane that the $Z(B_i), Z(C_i)$ can lie:

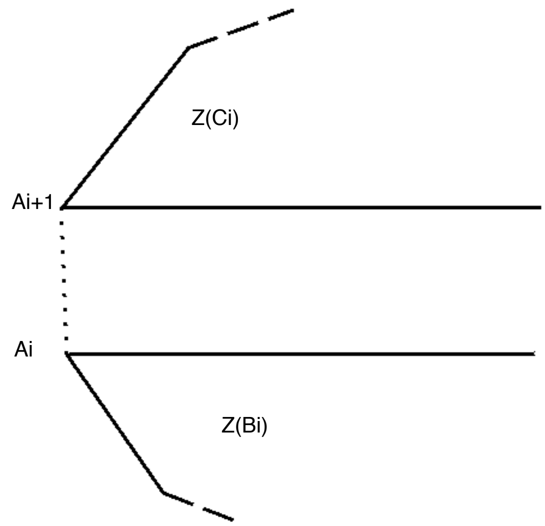

FIGURE 3. The regions in which $Z(B_i)$ and $Z(C_i)$ may lie.

Now, by the short exact sequence

$$0 \to B_i \to A_i \oplus A_{i+1} \to C_i \to 0$$

we get

$$Z(B_i) + Z(A_{i+1}) = Z(A_i) + Z(C_i)$$

Taking imaginary parts, and comparing with the inequalities 3, we get that equality must actually hold! Hence, $B_i = A_i$ and $C_i = A_{i+1}$.

For part b, we note that the (tangent of the) phase of A_{i+1}/A_i is the slope of the segment connecting $Z(A_i)$ and $Z(A_{i+1})$. As the region $\mathcal{H}$ is convex, this corresponds to consecutive sides having decreasing slopes. We can see this graphically too as in Figure 4.

For part c, assume that there exists A such that

$$\varphi(A) > \varphi(A_{i+1}/A_i) \text{ and } A \subset A_{i+1}/A_i$$

Look at the fibered square:

$$\begin{array}{ccc} \widehat{A} & \longrightarrow & A_{i+1} \\ \downarrow & & \downarrow \\ A & \longrightarrow & A_{i+1}/A_i \end{array}$$

which implies

$$Z(\widehat{A}) + Z(A_{i+1}/A_i) = Z(A_{i+1}) + Z(A)$$

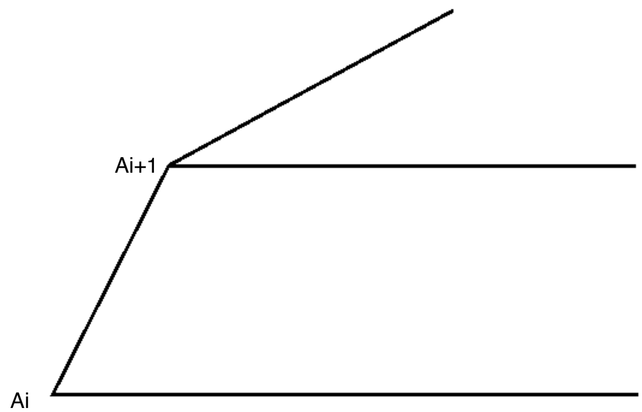

FIGURE 4. Decreasing slopes

which gives

$$Z(\widehat{A}) = Z(A) + Z(A_i)$$

As $\varphi(A) > \varphi(A_{i+1}/A_i)$, we conclude that $Z(\widehat{A}) \notin \mathcal{H}(E)$. However, $\widehat{A} \subset A_{i+1} \subset E$, hence by definition $Z(\widehat{A}) \notin \mathcal{H}(E)$. Contradiction! $\square$