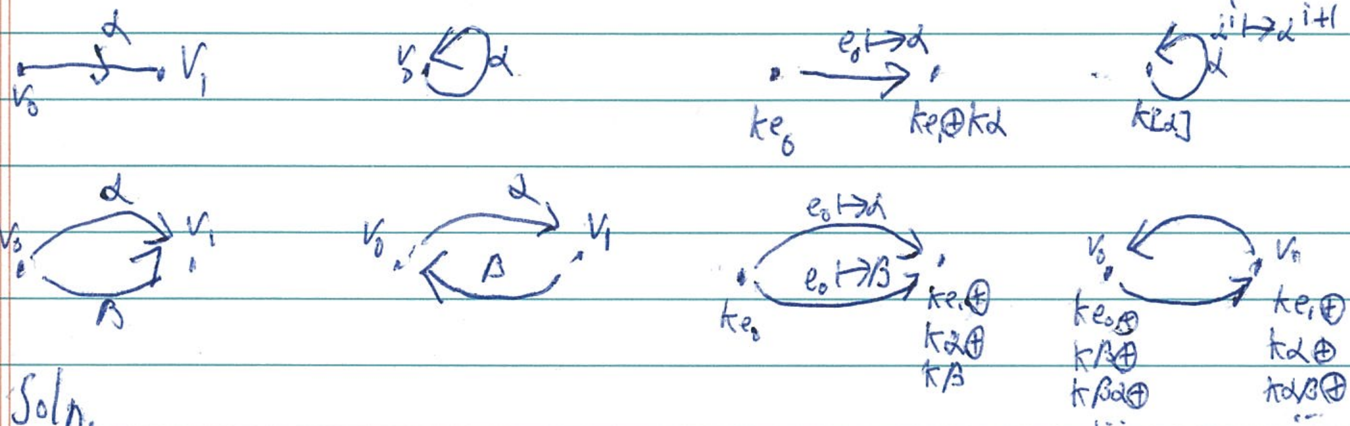
Bridgeland Stability Conditions Sept. 24

Recall: A quiver Q is a pair of sets Q_0 (vertices) and Q_1 (edges) with maps $s, t: Q_1 \to Q_0$ (source and target).

kQ is the free associative nonunital algebra gen'd by paths modulo concatenation.

Prop. $\mathrm{Rep}^{f.d.}(Q) \cong kQ\text{-}\underline{f.d.}\text{mod}$

Ex. Find the quiver representation corresponding to kQ as a kQ-module.



Soln.

Pf. of Prop. Given M, set $M \cong \bigoplus_{i \in Q_0} V_i$ with $V_i = e_i M$

Given a rep. set $M \cong \bigoplus_{i \in Q_0} V_i$. //

Def. Let $\mathcal{C}$ abelian. An object E is simple if any $E' \subset E$ has $E' = 0$ or $E' = E$.

Ex. For any Q, let $S(i)$ be the representation with k at vertex i and 0 elsewhere.

Ex. For $Q = \cdot \rightleftarrows \cdot$ $\quad \begin{tikzcd} k \arrow[r, bend left, "1"] & k \arrow[l, bend left, "1"] \end{tikzcd}$ is simple.

Prop. If Q is finite acyclic, any f.d. simple rep. is of the form $S(i)$ for some vertex i.

Pf. If Q has no cycles, then Q_0 is a poset, $i \leq j$ iff $\exists$ a directed path from i to j.

Given any rep. V, let i be maximal s.t. $V_i \neq 0$. Then $S(i) \hookrightarrow V$. //

Def. Let $\mathcal{C}$ abelian, $E \in \mathrm{Ob}\,\mathcal{C}$. A Jordan-Hölder filtration for E is a finite sequence $0 = E_0 \subset E_1 \subset \cdots \subset E_n = E$ such that E_{i+1}/E_i is simple $\forall i$.

Rmk. Unlike Harder-Narasimhan filtration, the J-H filtration is not unique. The simple factors are, however, unique up to reordering.

$\forall i$
$$\begin{tikzcd} E_0 \arrow[r] \arrow[d] & E_1 \arrow[r] \arrow[d] & E_2 \arrow[r] \arrow[d] & \cdots \arrow[r] & E \\ 0 \arrow[r] & E_1/E_0 \arrow[r] & E_2/E_1 & & \end{tikzcd}$$

Rmk. If every $E \in \mathrm{ob}\,\mathcal{C}$ admits a J-H filtration, $K_0(\mathcal{C})$ is spanned by simple objects.

Def. An abelian category $\mathcal{C}$ is called finite length if every object admits a J-H filtration.

Prop. Given an associative algebra A, the category $A\text{-}\mathrm{mod}^{\text{f.d.}}$ is finite length.

Pf. Given E, WLOG, E is not simple, or we're done. Then by induc on dimension, there is a simple subobject $S_0 = E_0$. Consider E/S_0. It has smaller dimension than E, so by induction on the dimension, it admits a Jordan-Hölder filtration. Pulling back the filtration to E and adding E_0 to the beginning, we get a filtration $0 \subset E_0 \subset E_1 \subset \cdots \subset E_n = E$. For $i \geq 0$, E_{i+1}/E_i is simple since it appears in the J-H filtration of E/E_0. E_0 is simple by assumption. //

Prop. If $\mathcal{C}$ is finite length, then $K_0(\mathcal{C}) \xleftarrow{\sim} \mathbb{Z}^{\mathrm{simpl}(\mathcal{C})}$

Pf. We know $\mathbb{Z}^{\mathrm{simpl}(\mathcal{C})} \twoheadrightarrow K_0(\mathcal{C})$ by finite length. Now define a map $K_0(\mathcal{C}) \to \mathbb{Z}^{\mathrm{simpl}(\mathcal{C})}$ $E \mapsto$ J-H factors. //

Cor. If Q is an acyclic quiver, $K_0(\mathrm{Rep}^{\text{f.d.}}(Q)) \cong \mathbb{Z}^{Q_0}$.

Recall: A Bridgeland stability cond. on $\overset{\text{abelian}}{\mathcal{C}}$ is a map $Z: \mathrm{ob}\,\mathcal{C} \to \mathbb{C}$ s.t.

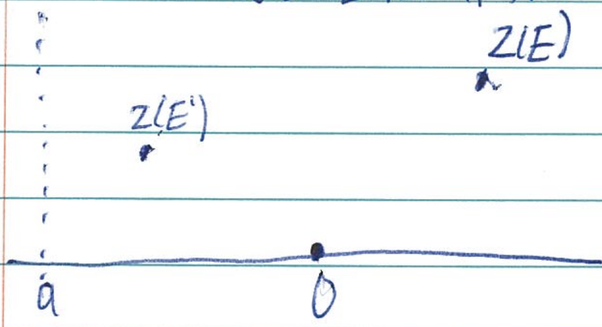
(0) $\mathrm{im}(Z) \subset \{re^{i\pi\theta} : \theta \in (0,1]\}$

(1) $Z(E) = 0 \Rightarrow E \cong 0$

(2) Z gives $K_0(\mathcal{C}) \to \mathbb{C}$ a homomorphism.

(3) Every object has a H-N filtration, ie,
$0 = E_0 \subset E_1 \subset \cdots \subset E_n = E$ where E_{i+1}/E_i is Z-semistable and
$\phi(E_{i+1}/E_i) > \phi(E_{i+2}/E_{i+1})$

Thm. Assume Z satisfies (0)-(2) and $\forall E \in \mathrm{ob}\,\mathcal{C}\ \exists a \in \mathbb{R}$ s.t.
$E' \subset E \Rightarrow \mathrm{Re}(Z(E')) \geq a$. Also assume $\mathcal{C}$ is finite length.



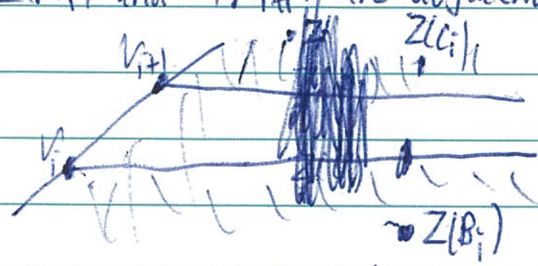
Then Z satisfies (3).

Pf. Fix E. Consider the set $\{Z(A) : 0 \overset{?}{\neq} A \subset E\}$. Let $\mathcal{H}(E)$ be the convex hull. By assumption, $\mathcal{H}(E)$ lies to the right of $\{\mathrm{Re}\,z = a\}$. By positivity and additivity, it's under $\{\mathrm{Im}\,z = \mathrm{Im}\,Z(E)\}$.

The set $\{Z(A)\}$ is discrete, because it's generated by finite positive linear combinations of finitely many objects. Then the set $\{\vec{v}_i \mid v_i$ is an extremal point of $\mathcal{H}(E)\}$ has finitely many $\vec{v}_i$ w/ $\arg(\vec{v}_i) > \arg(Z(E))$.

Lemma: Fix $A_i \subset E$ s.t. $Z(A_i) = v_i$. Then (a) $A_i \subset A_{i+1}$
(b) $\mathrm{Arg}(A_{i+2}/A_{i+1}) < \mathrm{Arg}(A_{i+1}/A_i)$
(c) A_{i+1}/A_i is semistable.

Pf. (a): Consider $B_i = A_i \wedge A_{i+1}$. Let $C_i = \mathrm{im}(A_{i+1} \oplus A_i) \to E$.
$Z(A_i)$ and $Z(A_{i+1})$ are adjacent vertices of $\mathcal{H}(E)$.



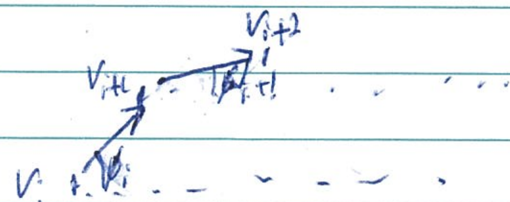
because v_i, v_{i+1} are extremal.

$\mathrm{Im}(Z(B_i)) \leq \mathrm{Im}(Z(A_i))$ and $\mathrm{Im}(Z(C_i)) \geq \mathrm{Im}(Z(A_{i+1}))$

$0 \to B_i \to A_i \oplus A_{i+1} \to C_i \to 0$ tells us $Z(B_i) + Z(C_i) = v_i + v_{i+1}$, s
$Z(B_i) = v_i$.

$0 \to B_i \to A_i \to A_i/B_i \to 0$, but $Z(B_i) = Z(A_i)$, so $B_i = A_i$.

Pf. of (b):



$\phi_{i+1} < \phi_i$