BRIDGELAND STABILITY CONDITIONS
LECTURE 8

Throughout this lecture, $\mathcal{C}$ will denote an abelian category.

1. Artinian and Noetherian categories

Definition 1.1. An object $E \in ob\mathcal{C}$ is *Artinian* if any descending chain

$$\ldots \subset E_{i+1} \subset E_i \subset \ldots \subset E_0 = E$$

stabilizes.

An object E is *Noetherian* if any ascending chain

$$E_0 \subset E_1 \subset E_2 \subset \ldots, \text{ where } E_i \subset E \text{ for any } i$$

stabilizes.

Remark 1.2. E Noetherian is equivalent to every sequence of epimorphisms

$$E \to D_1 \to D_2 \to \ldots$$

stabilizes.

Definition 1.3. A category $\mathcal{C}$ is Artinian (Noetherian) if every object is.

Proposition 1.4. *$\mathcal{C}$ is finite length if and only if $\mathcal{C}$ is Artinian and Noetherian.*

Proof. This is similar to what we did last lecture. If E is Artinian, then we can find a simple object $S_0 \subset E$. Define $D_0 = E/S_0$. If D_0 is not simple, we find a simple subject $S_1 \subset D_0$. Define $D_1 = D_0/S_1$. And so on. We get a sequence $D_i = D_{i-1}/S_i$, which stabilizes since $\mathcal{C}$ is Noetherian.

Let $f_i : E \to D_i$. Then

$$\ker f_0 \subset \ker f_1 \subset \ldots \subset ker f_n \subset E$$

is a J-H filtration for E.

The other direction is left as an exercise. □

2. Stability conditions

Recall the definition of Bridgeland stability conditions:

(0) $image(Z) \subset \bar{\mathbb{H}} - \mathbb{R}_{>0}$
(1) $Z(E) = 0 \implies E = 0$
(2) Z descends to $K_0(\mathcal{C})$
(3) HN filtration

Theorem 2.1. *Let $\mathcal{C}$ be an abelian category. Suppose we have Z satisfying* (0), (1) *and* (2)*. And*

($\star$1) *There is no infinite sequence of monomorphisms*

$$\ldots E_{i+1} \subset \ldots \subset E_0 = E$$

such that $\varphi(E_{j+1}) > \varphi(E_j)$.

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($\star$2) *And, there is no infinite sequence of quotients*

$$E \to D_1 \to D_2 \to \dots$$

with $\varphi(D_{j+1}) < \varphi(D_j)$

Then Z *satisfies* (3).

A situation in which these constrained infinite chains stabilize is when *every* infinite chain stabilizes. We get the important corollary.

Corollary 2.2. *If* $\mathcal{C}$ *is finite length, every* Z *satisfying (0)-(2) also satisfies (3).*

In the proof of Theorem 2.1, the following notion will play a key role.

Definition 2.3. Given an object $E \in ob\mathcal{C}$, and a central charge Z, a *maximally destabilizing quotient* (mdq) of E is a quotient $E \twoheadrightarrow B$ such that for any epimorphism $E \twoheadrightarrow B'$,

(1) $\varphi(B') \geq \varphi(B)$, and
(2) $\varphi(B') = \varphi(B)$ only if there exists a factorization

$$\begin{array}{ccc} E & \twoheadrightarrow & B \\ \downarrow & \swarrow & \\ B' & & \end{array}$$

Remark 2.4. In class there was a discussion whether if a mdq exists, then it is unique. In the case of $\mathcal{C}$ being an abelian category (which is our case, anyways), this is true.

To see that, we can reduce to the case of R-modules, for some ring R. Then we only have one option for the map $f : B \to B'$. Given an element $b \in B$, lift it to E, and then map it down to B'. If this map f is well defined, then it is unique. It will be well defined if all points in a fiber of $E \to B$ map down to the same element in B'.

In any case, we will not use this remark below.

Example 2.5. If E is semi-stable, the identity map $id : E \to E$ is mdq.

Remark 2.6. If $E \twoheadrightarrow B$ is a mdq, then B is semistable (since any quotient of B is also a quotient of E).

Remark 2.7. If $E \twoheadrightarrow B$ is a mdq, then $\varphi(B) \leq \varphi(E)$, since we could take $id : E \to E = B'$.

Remark 2.8. In the definition, we can take B' to be semistable, by the properties ($\star$) on Theorem 2.1.

Remark 2.9. Pictorially, $Z(B)$ is the quotient of maximal length among the ones with minimal slope.

Lemma 2.10. *For any* $E \in ob\mathcal{C}$*, under hypothesis of theorem 1, mdqs exist.*

Proof. Let E be an object in $\mathcal{C}$. If E is semistable, we are done.

If E is not semistable, there exists A such that

$$0 \to A \to E \to E' \to 0$$

with $\varphi(A) > \varphi(E)$. Using the chain condition, we may take A to be semistable.

Claim 2.11. *If $E' \twoheadrightarrow B$ is a mdq for E', then the composition*

$$E \twoheadrightarrow E' \twoheadrightarrow B$$

is an mdq for E.

Using this claim, we finish the proof of our lemma. We can construct a sequence of epimorphisms

$$E \twoheadrightarrow E' \twoheadrightarrow E'' \twoheadrightarrow \dots$$

This has to stabilize by property ($\star$). The last term will be the mdq for E. □

Proof of Claim 2.11. Assume we have

$$E \twoheadrightarrow B' \text{ and } \varphi(B') \leq \varphi(B)$$

We want to show that $\varphi(B') = \varphi(B)$ and that there is a map $B \to B'$ making the diagram below commute.

$$\begin{array}{ccc} E & \longrightarrow & B \\ \downarrow & \swarrow & \\ B' & & \end{array}$$

Let us draw a diagram of the central charges.

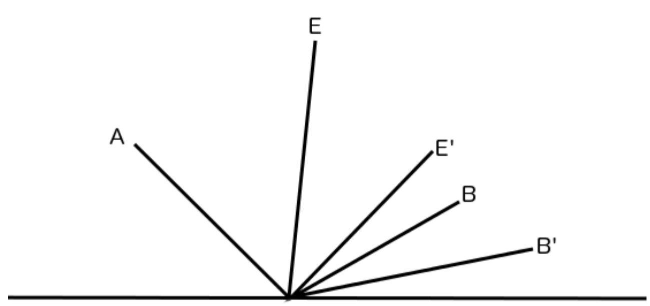


FIGURE 1. Central charges

We see that $\varphi(B') < \varphi(A)$. We can assume B' to be semi stable by remark 2.8. Since B', A are both s.s.,

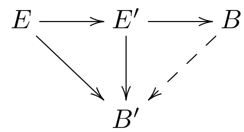
$$Hom(A, B') = 0$$

which implies that the composition $A \to E \to B'$ is the zero morphism.

$$\begin{array}{ccccc} A & \longrightarrow & E & \longrightarrow & E' \\ \downarrow & & \downarrow & \swarrow & \\ 0 & \longrightarrow & B' & & \end{array}$$

By the universal property of the cokernel E', there is a unique $E' \to B'$ factoring $E \to B$.

Since B is a mdq for E', we get $\varphi(B') \geq \varphi(B)$. Hence $\varphi(B') = \varphi(B)$. This in turn implies that there is a map $B \to B'$ which makes the diagram below commute, as we wanted to show.



□

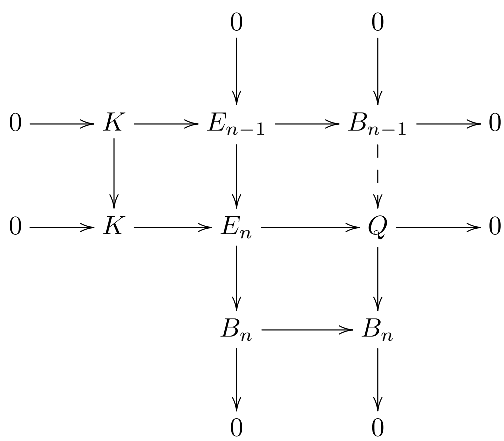
Proof of Theorem 2.1. Fix an object E. We want to produce a HN-filtration for it. If E is semistable, we are done. Otherwise, there exists

$$0 \to E_{n-1} \to E_n \to B_n \to 0$$

where $E_n = E$, and $E_n \twoheadrightarrow B_n$ being a mdq, B_n semistable and

$$\varphi(E_{n-1}) > \varphi(E_n)$$

Fix $E_{n-1} \twoheadrightarrow B_{n-1}$ an mdq. We can construct the following diagram:



We conclude that there is a sequence

$$0 \to B_{n-1} \to Q \to B_n \to 0$$

Lemma 2.12. $\varphi(B_{n-1}) > \varphi(B_n)$

Proof. It is enough to prove $\varphi(Q) > \varphi(B_n)$. Since B_n is mdq, we get $\varphi(Q) \geq \varphi(B_n)$. If $\varphi(Q) = \varphi(B_n)$, then they lie on the same ray, and B_n mdq implies

$$|Z(B_n)| \geq |Z(Q)|$$

On the other hand, the SES implies $|Z(Q)| \geq |Z(B_n)|$. Therefore, equality holds, and $Z(B_{n-1}) = 0$, which means $B_{n-1} = 0$. Contradiction! □

Now we produce an infinite descending chain

$$\ldots \subset E_{n-2} = K \subset E_{n-1} \subset E_n$$

which stabilizes by assumption $(\star)$. This is our HN-filtration. □

3. An example

We finish this lecture with an example that shows that there are stability conditions not detected by any of the two criteria we have seen in the past couple of lectures.

Example 3.1. Take

$$Cc = Coh(\mathbb{P}^1)^{op}$$

with the standard Z:

$$Z : ob\mathcal{C} \to \mathbb{C}$$
$$Z(E) = -\,deg(E) + irk(E)$$

does not satisfy the assumptions of the theorems, but is an stability condition.