Bridgeland Stability Conditions Thu, Sept, 26

Terminology: Let $\mathcal{C}$ abelian.
Def. $E \in \mathrm{ob}\,\mathcal{C}$ is artinian if any descending sequence of subobjects eventually stabilizes. E is noetherian if ascending sequences stabiliz

Rmk. E Noetherian iff every sequence of epis stabilizes.

Def. $\mathcal{C}$ is Artinian if every object is Artinian. Noetherian categori are defined analogously.

Prop. $\mathcal{C}$ is finite length iff it's Artinian + Noetherian.
Pf. Same as for f.d modules over an algebra.

E Artinian $\Rightarrow \exists S_0 \subset E$ simple. Define $D_0 = E/S_0$. If D_0 is not simple, find $S_1 \subset D_0$, S_1 simple. Define $D_1 = D_0/S_1$.

The sequence $D_i = D_{i-1}/S_i$ stabilizes by Noetherianeity. Let $f_i : E \to D_i$. $0 \subset \ker f_0 \subset \ker f_1 \subset \dots \subset \ker f_n \subset E$ will be a Jordan–Hölder filtration.

Def. A Bridgeland Stability condition has
(0) $\mathrm{im}(Z) = \{re^{i\theta}, \theta \in (0, \pi]\}$
(1) $Z^{-1}(\{0\}) = \{0\}$,
(2) Z descends to $K_0(\mathcal{C})$
(3) H-N filtration.

Thm. 1: $\mathcal{C}$ abelian, Z satisfies (0)-(2) and (*) $\nexists$ an infinite sequence of subobjects $\cdots \subset E_{j+1} \subset \cdots \subset E_0 = E$ such that $\phi(E_{j+1}) > \phi(E_j)$ and $\nexists$ infinite seq. of quotients $E \twoheadrightarrow D_1 \twoheadrightarrow D_2 \twoheadrightarrow \cdots$ w/ $\phi(D_{i+1}) < \phi(D_i)$. Then Z satisfies (3).

Cor. If $\mathcal{C}$ is finite length, and Z satisfies (0) - (2), then it satisfies (3).

Def. Given $E \in \mathrm{ob}\,\mathcal{C}$, a maximally destabilizing quotient (mdq) is a quotient $E \twoheadrightarrow B \neq 0$ such that

- $\forall E \twoheadrightarrow B'$, $\phi(B') \geq \phi(B)$.
- $\phi(B') = \phi(B)$ iff. $\exists$ factorization

$$\begin{tikzcd} E \arrow[r, two heads] \arrow[d] & B \\ B' \arrow[ur, dashed] & \end{tikzcd}$$

Ex. If E is semistable then $E \xrightarrow{id} E$ is mdq.

Rmk. If B is mdq, then B is semistable, since a quotient of B is a quotient of E.

Rmk. $\phi(B) \leq \phi(E)$, since $E \twoheadrightarrow E$ is an example of B'.

Rmk. In def'n, we can assume B' is semistable, by (*).

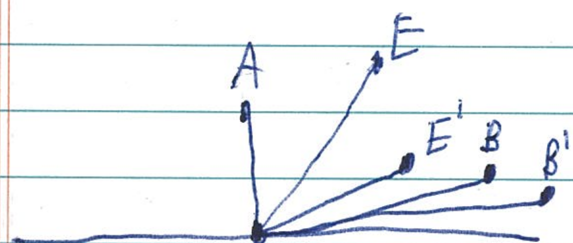
Lemma: If $E \in \operatorname{ob}\mathcal{C}$, under hyp. of thm. 1, mdq's exist.

Rmk. Pictorially, $Z(B)$ is the quotient w/ maximal length among those w/ minimal slope.

Pf. Assume WLOG E is not semistable, $0 \to A \to E \twoheadrightarrow E' \to 0$ w/ A semistable and $\phi(A) > \phi(E)$ (using chain condition to make A semistable).

Claim: If $E' \twoheadrightarrow B$ is an mdq for E', it's an mdq for E.

Pf. of claim: Assume we have $E \twoheadrightarrow B'$ with $\phi(B') \leq \phi(B)$.



$\phi(B') < \phi(A)$. We can assume B' to be semistable by one of the remark.
Since B', A are both s.s., $\mathrm{Hom}(A, B') = 0$.

$$\begin{tikzcd} A \arrow[r] \arrow[dd] & E \arrow[d] \arrow[dr, two heads] & \\ & E' \arrow[r] & B' \\ 0 \arrow[urr] & & \end{tikzcd}$$

commutes.

By the universal property of E', $\exists!\, E' \to B$, factoring $E \to B$.

Since B is an mdq for E', $\phi(B') \geq \phi(B)$ so $\phi(B') = \phi(B)$, so $\exists\, B \to B'$, but a map between semistable objects of the same phase is 0 or an isomorphism.

The process $E \twoheadrightarrow E' \twoheadrightarrow \cdots$ terminates by (*). //

Pf. of thm. 1: Fix $E \in \mathrm{Ob}\,\mathcal{C}$. If it's s.s., we're done. If E is not s.s., $\exists$ SES $0 \to E_{n-1} \to E \twoheadrightarrow B_n \to 0$ w/ B_n being mdq s.s. $\phi(E_{n-1}) > \phi(E)$.

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
 & K \arrow[r, equal] \arrow[d] & K \arrow[d] & & \\
0 \arrow[r] & E_{n-1} \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{② p.o.}"] & E_n \arrow[r] \arrow[d] & B_n \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & B_{n-1} \arrow[r, dashed, "c"] \arrow[d] & \exists\, Q \arrow[r] \arrow[d] & B_n \arrow[r] & 0 \\
 & 0 & 0 & &
\end{tikzcd}$$

Fix $E_{n-1} \twoheadrightarrow B_{n-1}$ mdq.

$$\begin{tikzcd}
K \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "A"] & E_{n-1} \arrow[r, hook] \arrow[d] \arrow[drr, phantom, "C"] & E_n \arrow[r, two heads] & Q \arrow[r] & 0 \\
0 \arrow[r, "\lrcorner"] & B_{n-1} \arrow[urr, dashed, "\exists!"'] & & &
\end{tikzcd}$$

B

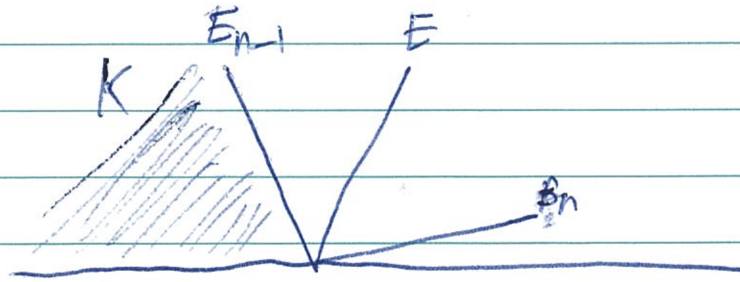
A, B p.o. $\Rightarrow$ C p.o.

$$\begin{tikzcd}
\cancel{E_{n-1}} \arrow[r] & \cancel{B_{n-1}} \\
\cancel{E_n} \arrow[r] & \cancel{Q}
\end{tikzcd}$$

$$\begin{tikzcd}
E_{n-1} \arrow[r] \arrow[d] & E_n \arrow[d] \\
B_{n-1} \arrow[r] \arrow[d] & Q \arrow[d] \\
Q \arrow[r] & B_n
\end{tikzcd}$$

diagram-chase.

Lemma: $\phi(Q) > \phi(B_n)$



$\phi(Q) \geq \phi(B_n)$ since B_n is an mdq.

⚠ If map in def'n of mdq is unique, then lemma follows.

$0 \to B_{n-1} \to Q \to B_n \to 0$ gives us $\phi(B_{n-1}) > \phi(B_n)$.

$$\begin{array}{c} E_{n-2} \\ \| \end{array}$$

$$\begin{tikzcd} K \arrow[r, hook] \arrow[d] & E_{n-1} \arrow[r, hook] \arrow[d] & E_n \arrow[d] \\ B_{n-2} & B_{n-1} & B_n \text{ s.s. since mdq.} \end{tikzcd}$$

By $(*)$, this eventually stabilizes.
$E_n \supset E_{n-1} \supset \cdots$ gives the HN filtration.

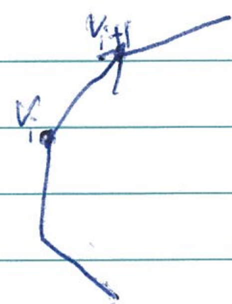
Lemma: $\phi(Q) > \phi(B_n)$.

Pf. of lemma: $\phi(Q) = \phi(B_n)$, they lie on same ray, and B_n mdq $\Rightarrow$
$|Z(B_n)| \geq |Z(Q)|$
SES $\Rightarrow |Z(Q)| \geq |Z(B_n)| \Rightarrow Z(B_{n-1}) \overset{\text{abs}}{=} 0 \Rightarrow B_{n-1} = 0$

Def. Fix $E \in \text{ob}\,\mathcal{C}$. $\mathcal{H}_Z(E) :=$ closed conv. hull of $\{Z(A) \mid A \subset E\}$.

Thm. 2: Let $\mathcal{C}$ abelian, Z satisfying (0)-(2) and $(**)$ $\forall E \in \text{ob}\,\mathcal{C}$, $\mathcal{H}_Z(E) \cap \{re^{i\pi\theta} \mid \theta \in (\phi(E), 1]\}$ is compact, and ∂ is a finite polygon.
Then Z satisfies (3).

Pf. of lemma part c: Need to show A_{i+1}/A_i semistable.



$Z(A_i) = v_i$.

Assume $\exists A$ s.t. $\phi(A) > \phi(A_{i+1}/A_i)$, $A \subset A_{i+1}/A_i$.

$$\begin{tikzcd} \hat{A} \arrow[r] \arrow[d] & A_{i+1} \arrow[d] \\ A \arrow[r, hook] & A_{i+1}/A_i \end{tikzcd}$$

$\Rightarrow Z(\hat{A}) + Z(A_{i+1}/A_i) = Z(A_{i+1}) + Z(A)$. $\Rightarrow$
$Z(A_{i+1}/A_i) = v_{i+1} - v_i$, $Z(A_{i+1}) = v_{i+1}$, so

$Z(\hat{A}) = Z(A) + v_i$