Stability conditions on triangulated categories:

Recall:

Thm 1: Let $\mathcal{C}$ be abelian. Let $Z : \text{Ob}\,\mathcal{C} \to \mathbb{C}$

Satisfy: (0) $\text{Im}(Z) \subseteq \overline{\mathbb{H}} \setminus \mathbb{R}^{>0}$
(1) $Z^{-1}(\{0\}) = \{0\}$
(2) Z induces a map on $K_0(\mathcal{C})$

Furthermore assume: $*$ Z is Artinian
and
$*$ Z is Noetherian

then Z satisfies (3) (H-N property)

Cor Let Q be a finite and acyclic Quiver.

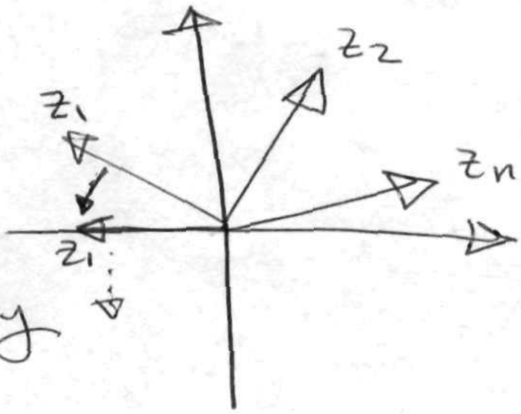
Then $\forall$ choice $\vec{Z} = (z_i) \in (\overline{\mathbb{H}} \setminus \mathbb{R}_{\geq 0})^{Q_0} \subseteq \mathbb{C}^{Q_0}$

the function $\text{ob}\,\text{Rep}^{f.d.}(Q) \to \mathbb{C}$

$V \to \sum z_i \dim(V_i)$

is a stability condition.

Question: Does it make sense to deform my stability condition past $\mathbb{R}_{<0}$ axis.



Answer: Yes, it defines a stability condition on $D^b(\text{Rep}^{f.d.}(Q))$

Basics of $D^b(\mathcal{C})$:

<u>Def</u> Given $\mathcal{C}$ abelian, a (co) chain complex is a sequence: $\cdots \to E^i \xrightarrow{d^i} E^{i+1} \xrightarrow{d^{i+1}} \cdots$ s.t. $d^{i+1} \circ d^i = 0$. The cohomology objects are $H^i(E) = \frac{\ker(d^i)}{\operatorname{im}(d^{i-1})}$

<u>Def</u> Given $\mathcal{C}$ abelian, the bounded derived category of $\mathcal{C}$, $D^b(\mathcal{C})$ is the category, where:

(I) Objects are chain complexes $E^\cdot$ s.t. $H^i(E^\cdot) \cong 0$ for all but finitely many i.

(II) Morphisms are homotopy classes of chain maps $E^\cdot \to F^\cdot$. (after resolving E or F)

i.e. $[f_0] = [f_1] \in \hom_{D^b(\mathcal{C})}(E, F)$ iff $\exists\, h$ s.t.

$dH + Hd = f_1 - f_0$

(III) Any morphism f inducing an $\cong$ on $H^\cdot$ has an inverse.

<u>Rmk</u> Let $0 \to A \to B \to C \to 0$ SES in $\mathcal{C}$.

Then there is a map $C \to A[1]$ (as chain complexes) where

$$A = \begin{array}{cl} 0 & \text{in } 1 \\ \uparrow & \\ A & \text{in } 0 \\ \uparrow & \\ 0 & \text{in } -1 \end{array} \quad \text{and} \quad A[1] = \begin{array}{cl} 0 & \text{in } 1 \\ \uparrow & \\ 0 & \text{in } 0 \\ \uparrow & \\ A & \text{in } -1 \end{array}$$

In genereal $A[i]_j := A_{i+j}$

We can replace
$$\begin{tikzcd} 0 \\ C \arrow[u] \\ 0 \arrow[u] \end{tikzcd}$$
by
$$\begin{tikzcd} 0 \\ B \arrow[u] \\ A \arrow[u] \\ 0 \arrow[u] \end{tikzcd}$$
. Then the map $C \to A[1]$

is equivalent to
$$\begin{tikzcd} 0 & 0 \\ 0 \arrow[u] & 0 \arrow[u] \\ B \arrow[u] \arrow[r] & 0 \arrow[u] \\ A \arrow[u] \arrow[r] & A \arrow[u] \\ 0 \arrow[u] & 0 \arrow[u] \end{tikzcd}$$
.

- Properties of $D^b(\mathcal{C}) = D$:

(I) $T : D \to D$ is an auto-equivalence.

$\qquad E \to E[1]$

(II) $\exists$ notion of <u>exact</u> or <u>distinguished</u> <u>triangles</u> ($A \xrightarrow{j} B \to C$

$$\begin{tikzcd} A \arrow[r] & B \arrow[dl] \\ C \arrow[u, dashed, "{[1]}"] & \end{tikzcd}$$
) with the

C is the "mapping cone" of j.

following properties:

(TR 1) •
$$\begin{tikzcd} X \arrow[r, "id"] & X \arrow[dl] \\ 0 \arrow[u, dotted] & \end{tikzcd}$$
is a distinguished triangle.

• $\forall \ X \to Y$, $\exists$
$$\begin{tikzcd} X \arrow[r] & Y \arrow[dl] \\ Z \arrow[u, dotted] & \end{tikzcd}$$

(TR 2) If
$$\begin{tikzcd} X \arrow[r] & Y \arrow[dl] \\ Z \arrow[u, dotted] & \end{tikzcd}$$
is an ex. tri, then
$$\begin{tikzcd} Y \arrow[r] & Z \arrow[dl] \\ X[1] \arrow[u, dotted] & \end{tikzcd}$$
and
$$\begin{tikzcd} Z[-1] \arrow[r] & X \arrow[dl] \\ Y \arrow[u, dotted] & \end{tikzcd}$$
are

exact triangles.

(TR3), (TR4) are omitted.

<u>"Defn"</u> A triangulated category D is an additive category with choices of Δ's satisfying : (I), (II)(TR 1) - (II)(TR4)

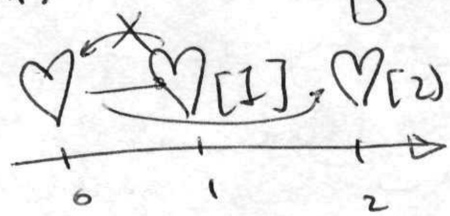
Rmk Assume $\mathcal{D} = D^b(\mathcal{C}) \supseteq \mathcal{C}$, $\mathcal{D}' = D^b(\mathcal{C}') \supseteq \mathcal{C}'$ and assume we're given an exact functor $F : \mathcal{D} \to \mathcal{D}'$ that's an equivalence.

What does $F(\mathcal{C}) \subseteq \mathcal{D}'$ look like?

It'll be <u>heart</u> for $\mathcal{D}$.

Defn A heart for a (bounded t-str on) triangulated category $\mathcal{D}$ is a full, additive subcategory $\heartsuit \subseteq \mathcal{D}$ s.t.

(H1) $\forall$ integers $k_1 > k_2$, $\hom_{\mathcal{D}}(E_1, E_2) = 0$ if $E_i \in \heartsuit[k_i]$



(H2) $\forall\ E \in \operatorname{ob} \mathcal{D}$, $\exists$ integers $k_1 > k_2 > \dots > k_n$ and a sequence of exact triangles:

$$\begin{tikzcd}
0 = E_0 \arrow[r] & E_1 \arrow[r] \arrow[d] & E_2 \arrow[r] \arrow[d] & \cdots \arrow[r] & E_{n-1} \arrow[r] & E_n = E \arrow[d] \\
& A_1 \arrow[ul, dashed] & A_2 \arrow[ul, dashed] & & & A_n \arrow[ul, dashed]
\end{tikzcd}$$

s.t. $A_i \in \heartsuit[k_i]$

Rmk By (H1) if $E, F \in ob(\heartsuit)$ then $Ext^{\leq -1}(E,F) \cong 0$

i.e. $Ext^n(E,F) \cong 0 \ \forall\, n \leq -1$.

b/c $Ext^{b-a}(E,F) \cong Ext^0(E[a], E[b])$ [It implies that $\heartsuit$ of any stable ∞-cat will be a discrete category.

Rmk $\bigcup_{k \in \mathbb{Z}} ob\, \heartsuit[k] \neq ob(\mathbb{D})$

Rmk Any object of $\mathcal{D}^b(\mathcal{C})$ is quasi-isomorphism to a bounded complex. (Not just bounded $H^\cdot$)

Pf $\forall\, b \in \mathbb{Z}$, let $\tau^{\leq b} E^\cdot$ be the chain complex

$$\to \cdots \to E^i \to E^{i+1} \to \cdots \to E^b \to \text{image}(d^b) \to 0 \to 0 \to 0 \to \cdots$$

there's a map $\tau^{\leq b} E^\cdot \to E^\cdot$

$$\begin{tikzcd}
\cdots \arrow[r] & E^i \arrow[r] \arrow[d, "id"] & \cdots \arrow[r] & E^b \arrow[r] \arrow[d, "id"] & \text{image}(d^b) \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d, "0"] & \cdots \\
\cdots \arrow[r] & E^i \arrow[r] & \cdots \arrow[r] & E^b \arrow[r] & E^{b+1} \arrow[r] & E^{b+2} \arrow[r] & \cdots
\end{tikzcd}$$

$\cong$ on $H^{\leq b}$, zero on $H^{>b}$.

Similarly let $\tau^{\geq a} E^\cdot$ be

$$\begin{tikzcd}
\cdots \arrow[r] & E^{a-2} \arrow[r] \arrow[d] & E^{a-1} \arrow[r, "d^{a-1}"] \arrow[d] & E^a \arrow[r] \arrow[d, "id"] & E^{a+1} \arrow[d, "id"] \\
\cdots & 0 \arrow[r] & \text{image}(d^{a-1}) \arrow[r] & E^a \arrow[r] & E^{a+1} \arrow[r] & \cdots
\end{tikzcd}$$

$\exists$ map $\tau^{\geq a} \leftarrow E^\cdot$

$\cong$ on $H^{\geq a}$ and 0 on $H^{<a}$.

$\underline{\underline{\text{Prop}^n}}$ $\mathcal{C}$ is a heart for $D^b(\mathcal{E})$.

$\underline{\underline{\text{Prop}^n}}$ A heart is always an abelian category.

$\underline{\underline{\text{Rmk}}}$ A map $j: A \to B$ in a heart is mono iff the triangle $A \to B \to C$ is also in the heart.

$\underline{\underline{\text{Def}^n}}$ The Grothendieck group of a triangulated cat D is the free abelian group generated by ob D, modulo the relation

$$\begin{tikzcd} A \arrow[r] & B \arrow[dl] \\ C \arrow[u] & \end{tikzcd} \Rightarrow \begin{array}{c}[A]+[B] \\ \shortparallel \\ [C]\end{array}$$

and is denoted by $K_0(D)$.

$\underline{\underline{\text{Prop}^n}}$: For any heart $K_0(D) \cong K_0(\heartsuit)$.

What should a stab condition on a triangulated category be?

We should have: (a) a function $Z: \mathrm{ob}(D) \to \mathbb{C}$ respecting exact triangles.

(b) Some ordering on (some) objects.

(c) A notion of semi-stable objects.

(d) A Harder-Narasimhan property.

(b): Doesn't make sense to order all objects only for semi-stable.

In abelian case, ordering was given by $\phi(E) = \arg(Z(E))$

But here $Z(E \oplus E[1]) = 0$ So this has no phase.

Not "indecomposable"

(c) Semistability can no longer be a property based only on Z. We need extra data.

<u>Ex</u> If we have $\mathcal{D} = D^b(\mathcal{C})$ then we'd have natural/naive choices for s.s objects:

$E \in ob(D^b(\mathcal{C}))$ is semi-stable $\Longleftrightarrow$ ~~$E \in ob\, \mathcal{C}$ is semi-stable~~ $E = E'[n]$ where $E' \in ob(\mathcal{C})$ is semi-stable.

This depends on the choice of heart.

On the other hand, without a heart we lost some information, because Z alone can't determine an ordering on ob $\mathcal{D}$. So we should just declare which objects are s.s.

Def A BSC on $\mathcal{D}$ is a pair $(Z, \heartsuit)$ where $\heartsuit$ is a heart and Z is a BSC on $\heartsuit$.

Defn (2) A BSC on $\mathcal{D}$ is a pair $(Z, \mathcal{P})$ where

- $Z : K_0(\mathcal{D}) \to \mathbb{C}$ is additive.
- $\mathcal{P}$ is a slicing ~~s.t. $Z(E) = me^{i\pi\phi}$~~ ~~where $m>0$ $E \in \mathcal{P}(\phi)$~~ such that for $E \in \mathcal{P}(\phi)$ we have $Z(E) = me^{i\pi\phi}$ for $m > 0$.

Prop^{n} (Next time) Def^{n}(1) $\iff$ Def^{n}(2)