Basics of $D^b(\mathcal{C})$

<u>Def</u>: Given an abelian category $\mathcal{C}$, a (co)chain complex is a sequence

$$\cdots \xrightarrow{d^{i-1}} E^i \xrightarrow{d^i} E^{i+1} \xrightarrow{d^{i+1}} \cdots$$

such that $d^{i+1}d^i = 0$ for all i. The cohomology objects are then $H^i(E^\circ) = {}^{\ker d^i}/_{\operatorname{Im} d^{i-1}}$.

<u>Def</u>: Given an abelian category $\mathcal{C}$, the bounded derived category of $\mathcal{C}$, written $D^b(\mathcal{C})$, is the category where,

- Objects are chain complexes E° such that $H^i(E^\circ) = 0$ for all but finitely many i.
- Morphisms are, (after resolving E° or F°), homotopy classes of chain maps
$$E^\circ \longrightarrow F^\circ$$
i.e., $[f_0] = [f_1] \in \hom_{D^b(\mathcal{C})}(E^\circ, F^\circ)$ iff $\exists H$ s.t.
$$dH + Hd = f_1 - f_0.$$
- Any morphism f inducing an isomorphism on all cohomology groups has an inverse.

<u>Remark</u>: Let $0 \to A \xrightarrow{j} B \to C \to 0$ be a short exact sequence in $\mathcal{C}$. Then there is a map $C \longrightarrow A[1]$ in $D^b(\mathcal{C})$.

<u>Pf</u>: We can replace $C = \begin{matrix} 0 \\ \uparrow \\ C \\ \uparrow \\ 0 \end{matrix}$ by $\begin{matrix} 0 \\ \uparrow \\ B \\ \uparrow j \\ A \\ \uparrow \\ 0 \end{matrix}$ up to quasi-isomorphism

There is a map $\begin{matrix} 0 \\ \uparrow \\ B \\ \uparrow \end{matrix} \longrightarrow \begin{matrix} 0 \\ \uparrow \\ 0 \\ \uparrow \end{matrix} = A[1].$

Triangulated Categories

Let us axiomatize some properties of $\mathcal{D} = D^b(\mathcal{E})$.

(0) There exists a functor
$$T : \mathcal{D} \longrightarrow \mathcal{D}$$
$$E \longmapsto E[1]$$

which is an auto-equivalence

(1) There exists a notion of <u>exact</u> or <u>distinguished</u> triangles
$$\begin{tikzcd} A \arrow[rr, "j"] & & B \arrow[dl] \\ & C \arrow[ul, dashed] & \end{tikzcd}$$
where C is "the mapping cone of j."

For any map $X \longrightarrow Y$, there exists a distinguished triangle
$$\begin{tikzcd} X \arrow[rr] & & Y \arrow[dl] \\ & Z \arrow[ul, dashed] & \end{tikzcd}$$

(TR1)
$$\begin{tikzcd} X \arrow[rr, "id."] & & X \arrow[dl] \\ & 0 \arrow[ul, dashed] & \end{tikzcd}$$
is a distinguished triangle

(TR2) If
$$\begin{tikzcd} X \arrow[rr] & & Y \arrow[dl] \\ & Z \arrow[ul, dashed] & \end{tikzcd}$$
, then
$$\begin{tikzcd} Y \arrow[rr] & & Z \arrow[dl] \\ & X[1] \arrow[ul, dashed] & \end{tikzcd}$$
&
$$\begin{tikzcd} Z[-1] \arrow[rr] & & X \arrow[dl] \\ & Y \arrow[ul, dashed] & \end{tikzcd}$$
.

(TR3), (TR4) More complicated. We will discuss them next time

<u>Def</u>: A triangulated category $\mathcal{D}$ is an additive category with a choice of distinguished triangles satisfying (0) + (TR1-TR4

Hearts

When $D = D^b(\mathcal{C})$, one has the structure of a distinguished subcategory $\mathcal{C} \subset D$. Let us axiomatize the properties of this subcategory.

Def: A heart for a (bounded T-structure on a) triangulated category D is a full additive subcategory $\heartsuit \subset D$ such that

(H1) For all integers $k_1 > k_2$,

$$\hom_D(E_1, E_2) = 0$$

if $E_i \in \heartsuit[k_i]$.

(H2) For all objects $E \in \operatorname{ob} D$, $\exists$ integers

$$k_1 > k_2 > \ldots > k_n$$

and a sequence of exact triangles

$$\begin{tikzcd}
0=E_0 \arrow[rr] & & E_1 \arrow[rr] \arrow[dl] & & E_2 \arrow[r] \arrow[dl] & \cdots \arrow[r] & E_{n-1} \arrow[rr] & & E_n = E \arrow[dl] \\
& A_1 \arrow[ul, dashed] & & A_2 \arrow[ul, dashed] & & & & A_n \arrow[ul, dashed] &
\end{tikzcd}$$

such that $A_i \in \heartsuit[k_i]$

Remark: By (H1), if $E, F \in \operatorname{ob}(\heartsuit)$, then

$$\operatorname{Ext}^n(E,F) \cong 0 \quad \forall n \leq -1$$

In other words, the $\heartsuit$ of any stable ∞-category will be a discrete category

Remark: $\bigcup_{k \in \mathbb{Z}} \text{ob} \heartsuit[k] \neq \text{ob} D$

Remark: Any object of $D^b(\mathcal{C})$ is quasi-isomorphic to a bounded complex (not merely one with bounded H°)

Pf: For all $b \in \mathbb{Z}$, let $\tau^{\leq b} E^\circ$ be the chain complex

$$\cdots \to E^i \to \cdots \to E^b \to \text{image}(d^b) \to 0 \to 0 \to \cdots$$

There is a natural map $\tau^{\leq b} E^\circ \longrightarrow E^\circ$ which is an iso on $H^{\leq b}$ and zero on $H^{>b}$.
There is an analogous cutoff complex $\tau^{\geq a} E^\circ$ which receives a map from E° that is an isomorphism on $H^{\geq a}$ and 0 on $H^{<a}$.
We obtain the result by choosing b sufficiently large and a sufficiently small.

Prop: $\mathcal{C}$ is a heart for $D^b(\mathcal{C})$.

Prop: A heart is always an abelian category.
A map $j: A \to B$ in a heart is a mono if and only if the triangle $A \to B \to C$ is also in the heart.

Def: The Grothendieck group of a triangulated category D is the free abelian group generated by ob D, modulo the relations

$$\begin{array}{ccc} A & \to & B \\ & \nwarrow \; \swarrow & \\ & C & \end{array} \Longrightarrow [A] + [C] = [B]$$

Prop: For any heart, $K_0(D) \cong K_0(\heartsuit)$.

Stability conditions on triangulated categories

Anything worthy of being called a Bridgeland Stability Condition on $\mathcal{D}$ should contain the data of:

(a) A function $Z: \mathrm{ob}\,\mathcal{D} \to \mathbb{C}$ that respects exact triangles.

(b) A notion of semi-stable objects.

(c) An ordering ϕ on (at least) the semi-stable objects

(d) A Härder-Narasimhan property

Unfortunately, unlike in the case of an abelian category semi-stability can not be a property based only on Z. This is because $Z(E \oplus E[1]) = 0$ for any object E. Semi-stability must be explained in the form of additional data. There are two approaches, which we will see to be equivalent:

<u>Def 1</u>: A BSC on a triangulated category $\mathcal{D}$ is a pair $(Z, \heartsuit)$ where $\heartsuit$ is a heart for $\mathcal{D}$ and Z is a BSC on $\heartsuit$.

<u>Def 2</u>: A BSC on $\mathcal{D}$ is a pair (Z, P) where $Z: K_0(\mathcal{D}) \to \mathbb{C}$ is additive & P is a slicing such that $Z(E) = m e^{i\pi\phi}$ where $m > 0$ & $E \in P(\phi)$.