Stability Conditions on Triangulated Categories, Revisited.

Ali + Jeremy taking notes

Recall from last time:

<u>Thm 1</u>. Let $\mathcal{C}$ be abelian.

Let $Z : \mathrm{ob}\,\mathcal{C} \to \mathbb{C}$ be a function s.t.

(0) $\mathrm{image}(Z) \subset \overline{\mathbb{H}} \setminus \mathbb{R}_{\geq 0}$

(1) $Z(E) = 0 \Rightarrow E \cong 0$

(2) Z induces a map on $K_0(\mathcal{C})$.

Further assume

* Z-Artinian condition
* Z-Noetherian condition.

Then Z satisfies the Harder-Narasimhan property.

Cor. Let Q be a finite, acyclic quiver. Then $\forall$ choices

$$\vec{Z} \in \left(\overline{\mathbb{H}} \setminus \mathbb{R}_{\geq 0}\right)^{Q_0} \subset \mathbb{C}^{Q_0}$$

one obtains a stability condition

$$Z : \mathrm{Rep}^{f.d.}(Q) \longrightarrow \mathbb{C}$$

$$V \longmapsto \sum_{i \in Q_0} Z_i \dim V_i .$$

Pf. Since Q is acyclic,

$$K_0(Q) \cong \bigoplus_{i \in Q_0} [S(i)] \cong \mathbb{Z}^{Q_0} .$$

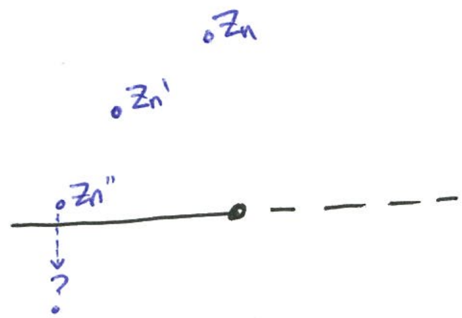
Since dimension respects SES, and $\dim (S(i))_j = \begin{cases} 1 & i = j \\ 0 & \text{otherwise} \end{cases}$

this map sends

$$V \longmapsto \left(\dim V_1, \ldots, \dim V_n\right) \in \mathbb{Z}^{Q_0} .$$

The only representation w/ $\dim V_i = 0 \ \forall i$ is the zero representation, so (1) is satisfied. (0) is satisfied since each Z_i is in $\overline{\mathbb{H}} \setminus \mathbb{R}_{\geq 0}$, and (2) is satisfied by the beginning of this proof. Since $\mathrm{Rep}^{f.d.}(Q)$ is finite length, the Z-Artinian / Z-Noetherian conditions are satisfied. //

But can we "cross" the bdry?



When we pass to $D^b(\mathrm{Rep}^{f.d.} Q)$, we'll discover a notion of stability condition that gives meaning to this.

<u>Rmk</u> If Q not acyclic we'll want $D^b(\mathrm{Rep}\, Q)$ – the bounded derived category of Q-representations w/ finite-dimensional cohomology.

LATER

What should a stability condition on some triangulated category be?

(a) We should have a stability function
$$\mathrm{ob}\, D \xrightarrow{Z} \mathbb{C}$$
respecting <u>exact triangles</u>.

the ~~idea of~~ analogue of a SES for triangulated categories.

(b) We should have a notion of <u>ordering</u> of objects.

(c) We should have a notion of semistable objects.

(d) Some sort of Harder-Narasimhan property.

Defn Given an abelian category $\mathcal{C}$, a cochain complex in $\mathcal{C}$ is a complex

$$\to E^i \xrightarrow{d^i} E^{i+1} \to \cdots$$

of objects $E^i \in \mathrm{ob}\,\mathcal{C}$ s.t. $d^{i+1} \circ d^i = 0$.

Its cohomology groups are $H^i(E^\bullet) := \ker(d^i) / \mathrm{image}(d^{i-1})$.

Defn $D^b(\mathcal{C})$ is the category where

- objects are cochain complexes $E^\bullet$ s.t. $H^i(E^\bullet) \cong 0$ for all but finitely many i. ("bounded")
- morphisms are homotopy classes of chain maps $f: E^\bullet \to F^\bullet$.

after an appropriate resolution of $E^\bullet$ or $F^\bullet$.

(So $[f_0] = [f_1] \in \hom(E^\bullet, F^\bullet)$ iff $\exists\, H$ s.t. $dH + Hd = f_1 - f_0$).

- any morphism f inducing an $\cong$ on $H^\bullet$ has an inverse.

NOT simple.

i.e. f is a quasi-isomorphism

"The bounded derived category of $\mathcal{C}$"

__Rmk__ Let $0 \to A \to B \to C \to 0$ be a SES in $\mathcal{C}$.

In $D^b(\mathcal{C})$, C is quasi isomorphic to

$$
\begin{array}{cccc}
1 & 0 & & 0 \\
 & \uparrow & & \uparrow \\
0 & B & \xrightarrow[f]{\sim} & C \\
 & \uparrow & & \uparrow \\
-1 & A & & 0 \\
 & \uparrow & & \\
-2 & 0 & & \\
 & & & \\
 & C' & &
\end{array}
$$

But this complex has a natural map to $A[1]$!

$$
\begin{array}{cccc}
-1 & 0 & & 0 \\
 & \uparrow & & \uparrow \\
0 & B & \longrightarrow & 0 \\
 & \uparrow & & \uparrow \\
-1 & A & & A \\
 & \uparrow & & \uparrow \\
-2 & 0 & & 0 \\
 & & & \\
 & C' & & A[1]
\end{array}
$$

This happens in general when $A^\cdot, B^\cdot$ are cochain complexes and $C^\cdot$ is a mapping cone, i.e., homotopy cokernel.

$$0 \longrightarrow A \xrightarrow{f} B \longrightarrow \mathrm{Cone}(f) \longrightarrow A[1]$$

for this reason, exact triangles are often drawn as

$$
\begin{tikzcd}
A \arrow[r] & B \arrow[dl] \\
C \arrow[u, dashed] &
\end{tikzcd}
\quad \text{or} \quad
\begin{tikzcd}
A \arrow[r] & B \arrow[dl] \\
C \arrow[u, "{[1]}"] &
\end{tikzcd}
$$

$D^b(\mathcal{C})$ can be equipped w/ data of

which sequences

$$A^\bullet \xrightarrow{f} B^\bullet \to C^\bullet$$

should be like ~~mapping cone~~ cofiber sequences, i.e.,

should exhibit $C^\bullet$ as like a mapping cone.

<u>Ex</u>
$$\begin{tikzcd}
A \arrow[rr, "f"] & & B \arrow[dl] \\
& \mathrm{Cone}(f) \arrow[ul, dashed] &
\end{tikzcd}$$

<u>Ex</u>
$$\begin{tikzcd}
A \arrow[rr] & & 0 \arrow[dl] \\
& A[1] \arrow[ul, dashed] &
\end{tikzcd}$$

$\wr$

$$\begin{tikzcd}
\vdots & \vdots & \vdots \\
A^1 & A^1 \oplus A^2 & A^2 \\
A^0 \arrow[u] \arrow[r, hook] & A^0 \oplus A^1 \arrow[u, "d"] \arrow[r, two heads] & A^1 \arrow[u] \\
A^{-1} \arrow[u] & A^{-1} \oplus A^0 \arrow[u, "d"] & A^0 \arrow[u] \\
\vdots & \vdots & \vdots \\
A \arrow[r, hook] & \overset{\simeq}{0} \arrow[r, two heads] & A[1]
\end{tikzcd}$$

Such sequences are called <u>exact</u> <u>triangles</u>,

and give $D^b(\mathcal{C})$ the structure of a <u>triangulated</u> <u>category</u>.

Specifically:

Properties of $D^b(\mathcal{C})$:

(0) $\exists$ a functor $T : D \longrightarrow D$

$$E \longmapsto E[1]$$

which is an autoequivalence, along w/ notion of "triangle," or "distinguished triangle".

(TR1) $\cdot$ $X \xrightarrow{id} X \longrightarrow 0$.

$$\begin{tikzcd} X \arrow[rr] & & X \arrow[dl] \\ & 0 \arrow[ul, dashed] & \end{tikzcd}$$

$\cdot$ $\forall$ $X \longrightarrow Y$, $\exists$ $X \longrightarrow Y \longrightarrow Z$

$$\begin{tikzcd} X \arrow[rr] & & Y \arrow[dl, "\exists"] \\ & Z \arrow[ul, dashed] & \end{tikzcd}$$

(mapping cones exist)

(TR 2)

$$\begin{tikzcd} X \arrow[rr] & & Y \arrow[dl] \\ & Z \arrow[ul, dashed] & \end{tikzcd}$$

$$\Downarrow$$

$$\begin{tikzcd} Y \arrow[rr] & & Z \arrow[dl] \\ & X[1] \arrow[ul, dashed] & \end{tikzcd} \qquad \begin{tikzcd} Z[-1] \arrow[rr] & & X \arrow[dl] \\ & Y \arrow[ul, dashed] & \end{tikzcd}$$

(You can rotate triangles)

(TR3) omitted

(TR4) omitted.

(TR3) and (TR4) follow as an awkward reformulation of homotopy cokernels.

(TR3) $\forall$ comm. diagrams
$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "h"'] \arrow[dr, phantom, "\circlearrowleft"] & Y \arrow[d] \\ X' \arrow[r, "f'"'] & Y' \end{tikzcd}$$
in D,

$\exists$ map
$$\begin{tikzcd} X \arrow[r] \arrow[d, "h"'] & Y \arrow[r] \arrow[d] & C(f) \arrow[r] \arrow[d, dashed] & X[1] \arrow[d, "h[1]"] \\ X' \arrow[r] & Y' \arrow[r] & C(f') \arrow[r] & X'[1] \end{tikzcd}$$

making above diagram ~~complete~~ commute.

__Ex__ *If $f: 0 \to Y'$, we have*
$$\begin{tikzcd} X \arrow[r] \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & Y \arrow[r] \arrow[d] & C(f) \arrow[r] \arrow[d, dashed, "\exists"'] & X[1] \arrow[d] \\ 0 \arrow[r] & Y' \arrow[r, "id"] & Y' \arrow[r] & 0 \end{tikzcd}$$

which *looks like univ. property of cokernels:*
$$\begin{tikzcd} X \arrow[r] \arrow[d] & Y \arrow[ddr, bend left] & \\ 0 \arrow[drr, bend right] & & \\ & & Y' \end{tikzcd} \implies \begin{tikzcd} X \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & Y \arrow[d] \arrow[ddr, bend left] & \\ 0 \arrow[r] \arrow[drr, bend right] & C(f) \arrow[dr, dashed, "\exists"] & \\ & & Y' \end{tikzcd}$$

But the diagram
$$\begin{tikzcd} X \arrow[r] \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & Y \arrow[d] \\ 0 \arrow[r] & Y' \end{tikzcd}$$
__forgets__ the 2-cell making diagram homotopy commutative

If one <u>retains</u> information of the homotopy

$$\begin{tikzcd} X \arrow[r] \arrow[d] \arrow[dr, phantom, "\swarrow"] & Y \arrow[d] \\ 0 \arrow[r] & Y' \end{tikzcd}$$

making diagram homotopy commutative, we have

$$\begin{tikzcd} X \arrow[r] \arrow[d] \arrow[dr, phantom, "\swarrow"] & Y \arrow[d] \arrow[ddr, bend left] & \\ 0 \arrow[r] \arrow[drr, bend right] & C(f) \arrow[dr, dashed] & \\ & & Y' \end{tikzcd}$$

unique up to contractible choice!

The possible non-uniqueness of (TR3) arrow comes from forgetting the homotopy.

More generally,

$$\begin{tikzcd} X \arrow[r] \arrow[d] \arrow[dr, phantom, "\swarrow"] & Y \arrow[d] \\ X' \arrow[r] & Y' \end{tikzcd} \rightsquigarrow \begin{tikzcd} & X \arrow[r, "f"] \arrow[dl] \arrow[dd, dashed] & Y \arrow[dl] \arrow[dd] \\ X' \arrow[r] \arrow[dd] & Y' \arrow[dd] & \\ & 0 \arrow[r] \arrow[dl] & C(f) \arrow[dl, dashed, "\exists!"] \\ 0 \arrow[r] & C(f') & \end{tikzcd}$$

up to contractible choice

by univ prop of homotopy cokernel.

Again, $\overbrace{\text{unique}}^{\text{(up to cont. choice)}}$ <u>only</u> if you remember the homotopy.

(TR4) says given triangles

$$
\begin{tikzcd} X \arrow[r, "x"] & Y \arrow[dl] \\ A \arrow[u, dashed] & \end{tikzcd}
,\quad
\begin{tikzcd} Y \arrow[r, "y"] & Z \arrow[dl] \\ C \arrow[u, dashed] & \end{tikzcd}
,\quad
\begin{tikzcd} X \arrow[r, "y\circ x"] & Z \arrow[dl] \\ B \arrow[u, dashed] & \end{tikzcd}
$$

then $\exists$ triangle

$$
\begin{tikzcd} A \arrow[r] & B \arrow[dl] \\ C \arrow[u, dashed] & \end{tikzcd}
$$

satisfying some properties.

This also follows from general ∞-categorical nonsense of homotopy colimits.

The 3 triangles should be thought of as

$$
\begin{tikzcd} X \arrow[r, "x"] \arrow[d] & Y \arrow[d] \arrow[dl, phantom, "\Swarrow"] \\ 0 \arrow[r] & A \end{tikzcd}
\qquad
\begin{tikzcd} X \arrow[rr, "\text{``}\simeq y\circ x\text{''}"] \arrow[d] & & Z \arrow[d] \\ 0 \arrow[rr] & & B \end{tikzcd}
\qquad
\begin{tikzcd} Y \arrow[r, "y"] \arrow[dd] & Z \arrow[dd] \\ & \\ 0 \arrow[r] & C \end{tikzcd}
$$

p.o. $\qquad$ p.o. $\qquad$ p.o.

We obtain

① $\begin{tikzcd} X \arrow[r] \arrow[d] & Y \arrow[r] \arrow[d] & Z \arrow[d] \\ 0 \arrow[r] & A \arrow[r] & B \end{tikzcd}$ $\xRightarrow{\text{by usual lemma about composing pushouts}}$ $\begin{tikzcd} Y \arrow[r] \arrow[d] & Z \arrow[d] \\ A \arrow[r] & B \end{tikzcd}$ p.o.

② $\begin{tikzcd} Y \arrow[r] \arrow[d] & Z \arrow[d] \\ A \arrow[r] \arrow[d] & B \arrow[d] \\ 0 \arrow[r] & C \end{tikzcd}$ $\xRightarrow{\text{same lemma}}$ $\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[d] \\ 0 \arrow[r] & C \end{tikzcd}$ p.o. This is the desired triangle.

Both hearts and slicings (together w/ Z) determine stability conditions. Let's start w/ hearts.

Defn A $\overbrace{\text{heart}}^{\text{(of a bounded t-structure)}}$ for a triangulated category D is

a full, additive subcategory

$$\heartsuit \subset D$$

Closed under finite (co)products of D itself!

such that

(H1) $\forall$ integers $k_1 > k_2$,

$$\hom_D(E_1, E_2) = 0$$

whenever $E_i \in \heartsuit[k_i]$.

(H2) $\forall$ $E \in \text{ob}\, D$, $\exists$ integers

$$k_1 > k_2 > \cdots > k_n$$

and a sequence of exact triangles

$$\begin{tikzcd}
0 = E_0 \arrow[rr] & & E_1 \arrow[rr] \arrow[dl] & & E_2 \arrow[r] \arrow[dl] & \cdots \arrow[r] & E^{n-1} \arrow[rr] & & E^n = \cancel{E} \arrow[dl] \\
& A_1 \arrow[ul, dashed] & & A_2 \arrow[ul, dashed] & & & & A_n \arrow[ul, dashed] &
\end{tikzcd}$$

such that

$$A_i \in \heartsuit[k_i].$$

__Rmk__ By (H1), if $E, F \in \mathrm{ob}\,\heartsuit$, then

$$\mathrm{Ext}^{\leq -1}(E,F) \cong 0.$$

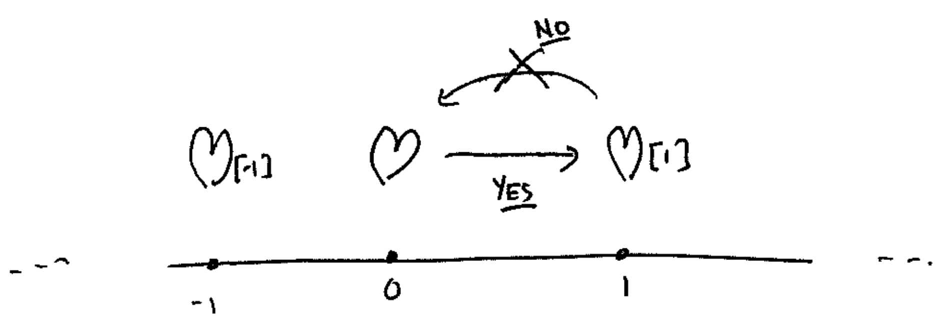
i.e.,

$$\mathrm{Ext}^n(E,F) \cong 0 \quad \forall \ n \leq -1.$$

This is because $\mathrm{Ext}^{b-a}(E,F) \cong \mathrm{Ext}(E[a], F[b])$.

So $\mathrm{Ext}^{-a}(E,F) \cong \mathrm{Ext}^0(E[a],F) \cong \hom_D(E[a],F) = 0 \quad \forall \ a \geq 1$.

Since $\mathrm{Ext}^{\leq 0}$ are <u>homotopy groups</u> by Dold-Kan, this really means (if D came from a dg or a stable ∞-category) that the hom-<u>space</u> is <u>discrete</u> up to homotopy equivalence.



__Rmk__

$$\bigcup_{k \in \mathbb{Z}} \mathrm{ob}\,\heartsuit[k] \neq \mathrm{ob}\,D.$$

Rather, objects of D are obtained via the <u>extension</u> <u>closure</u> of $\bigcup \mathrm{ob}\,\heartsuit[k]$.

Defn The Grothendieck group $K_0(\mathcal{D})$ of a triangulated category $\mathcal{D}$ is the free abelian group generated by ob $\mathcal{D}$, modulo the relation

$$\begin{tikzcd} A \arrow[r] & B \arrow[dl] \\ C \arrow[u, dashed] & \end{tikzcd} \quad \Longrightarrow \quad [A] + [C] = [B] \text{ in } K_0(\mathcal{D}).$$

Propn If $\heartsuit \subset \mathcal{D}$ is a heart,

$$K_0(\heartsuit) \cong K_0(\mathcal{D}).$$

Pf Some other time.

Rmk Any object $E^\bullet$ is quis to a bounded complex (not just something w/ bounded cohomology).

Pf. $\forall$ $b \in \mathbb{Z}$, let

$$\tau^{\leq b} E^\bullet$$

be the chain complex

$$\cdots \longrightarrow E^i \to E^{i+1} \to \cdots \longrightarrow E^b \to \mathrm{image}(d^b) \to 0.$$

$\exists$ a map

$$\tau^{\leq b} E^\bullet \longrightarrow E^\bullet$$

given by

$$\begin{tikzcd}
\cdots E^i \cdots \arrow[d, "id"] & \longrightarrow E^b \arrow[d, "id"'] \arrow[r] & \mathrm{image}(d^b) \arrow[d] \arrow[r] & 0 \arrow[d] \arrow[r] & 0 \\
\cdots E^i \cdots & \longrightarrow E^b \arrow[r] & E^{b+1} \arrow[r] & E^{b+2} \arrow[r] & \cdots
\end{tikzcd}$$

This is $\cong$ on $H^{\leq b}$, zero on $H^{>b}$.

$\forall$ $a \in \mathbb{Z}$, let

$$\tau^{\geq a} E^\bullet \longleftarrow E^\bullet$$

be

$$\begin{tikzcd}
\cdots \arrow[r] & E^{a-1} \arrow[d] \arrow[r] & E^a \arrow[d, "id"] \arrow[r] & \cdots \arrow[d, "id"] \\
0 \arrow[r] & \mathrm{image}(d^{a-1}) \arrow[r] & E^a \arrow[r] & \cdots
\end{tikzcd}$$

$\cong$ on $H^{\geq a}$,
0 on $H^{<a}$. //

Prop $\mathcal{C}$ is a heart for $D^b(\mathcal{C})$.

Pf. If $E^\bullet$ is some cochain complex, bounded, let

$$H^\bullet(E^\bullet) \cong H^a(E^\bullet) \oplus \cdots \oplus H^b(E^\bullet), \qquad a \le b \in \mathbb{Z}.$$

Set

$$\begin{tikzcd}
E_b = \arrow[d] & \cdots \arrow[r] & E^i \arrow[r, "d^i"] \arrow[d] & E^{i+1} \arrow[r] \arrow[d] & \cdots \arrow[r] & E^{b-1} \arrow[r] \arrow[d] & \mathrm{image}(d^{b-1}) \arrow[r] \arrow[d] & 0 \\
E & & E^i \arrow[r] & E^{i+1} & & E^{b-1} \arrow[r] & E^b \arrow[r] & 0.
\end{tikzcd}$$

By LES of cohomology,

$$H^\bullet(E/E_b) \cong H^b(E) \in \heartsuit[-b].$$

Now set

0 cohomology   0 cohomology

$$\begin{tikzcd}
E_{b-1} = \arrow[d] & \cdots \arrow[r] & E^i \arrow[r] \arrow[d] & E^{i+1} \arrow[r] \arrow[d] & \cdots \arrow[r] & E^{b-2} \arrow[r] \arrow[d] & \mathrm{image}(d^{b-2}) \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \cdots \arrow[d] \\
E_b = & & E^i \arrow[r] & E^{i+1} \arrow[r] & \cdots \arrow[r] & E^{b-2} \arrow[r] & E^{b-1} \arrow[r] & \mathrm{image}(d^{b-1}) \arrow[r] & 0 \cdots
\end{tikzcd}$$

H^{b-1} 0 cohomology

Again by LES,

$$H^\bullet(E_b/E_{b-1}) \cong H^{b-1}(E_b) \cong H^{b-1}(E) \in \heartsuit[-(b-1)].$$

Then set

$$\begin{tikzcd}
E_a = \arrow[d] & 0 \arrow[r] & \mathrm{Ker}(d^a) \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \arrow[d] & \cdots \\
E_{a+1} = & 0 \arrow[r] & E^a \arrow[r] & \mathrm{image}(d^a) \arrow[r] & 0 & \cdots
\end{tikzcd}$$

We thus have

$$\begin{tikzcd}
0 = E_0 \arrow[rr, bend left] & & E_{a+1} \arrow[r] \arrow[dl] & \cdots \arrow[r] & E_{b-1} \arrow[r] & E_b \arrow[r] \arrow[dr] & E_{b+1} = E \arrow[dl] \\
& H^a E \arrow[ul, dashed] & & & H^{b-1} E \arrow[u, dashed] \arrow[from=u, phantom] & H^b E \arrow[u, dashed] & \\
& \heartsuit[-a] \arrow[u, phantom, "\cap"] & & & \heartsuit[b-1] \arrow[u, phantom, "\cap"] & \heartsuit[-b] \arrow[u, phantom, "\cap"] &
\end{tikzcd}$$

Reindexing the subscripts we thus have the filtration of (H2), with

$$\cancel{\heartsuit = k\text{-mod} \subset \mathrm{Chain}^b_k},$$

$$\heartsuit = k\text{-mod} \subset D^b(k\text{-mod}).$$

__Rmk__ Same proof shows $\mathrm{Coh}(X)$ is a heart for $D^b\,\mathrm{Coh}(X)$.

bounded complexes of coherent sheaves

<u>Prop'n</u> For any triangulated $\mathcal{D}$, a heart is an abelian category.

<u>Pf</u> Some other time.

<u>Rmk</u>. Given a heart $\heartsuit \subset \mathcal{D}$, what are its SES?

We (find) say a morphism $A \to B$ is mono iff the triangle $A \to B \to C$ is contained in $\heartsuit$.

Likewise, we'll find that $B \to C$ is an epi iff $A \to B \to C$ is contained in $\heartsuit$.

<u>Ex</u> If $j: A \to B$ not injective, replace $A \xrightarrow{j} B$

by

$$\begin{tikzcd}
0 & 0 \\
A \arrow[u] \arrow[r, "{(id,j)}"] & A \oplus B \arrow[u] \\
0 \arrow[u] & A \arrow[u, "{(id,j)}"'] \\
0 \arrow[u] & 0 \arrow[u]
\end{tikzcd}$$

. The cone is

$$\begin{tikzcd}
0 \\
B \arrow[u] \\
A \arrow[u, "j"] \\
0 \arrow[u]
\end{tikzcd}$$

whose H^* is $H^0 \cong \operatorname{Coker} j$

$H^{-1} \cong \ker j$.

So the triangle $A \xrightarrow{j} B \to C$ is NOT contained in $\heartsuit$, since C isn't.