Interpretation of (TR3) and (TR4) in terms of mapping cones:

(TR3) Given a commutative diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & Y \arrow[d] \\ X' \arrow[r, "f'"] & Y' \end{tikzcd}$$

Using the axioms (TR1) and (TR2) we have

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & Y \arrow[r] \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & C(f) \arrow[r] \arrow[d, dashed] \arrow[dr, phantom, "\circlearrowleft"] & X[1] \arrow[d, "g[1]"] \\ X' \arrow[r, "f'"] & Y' \arrow[r] & C(f') \arrow[r] & X'[1] \end{tikzcd}$$

(TR3) states that there is $\begin{tikzcd} C(f) \arrow[d] \\ C(f') \end{tikzcd}$ such that all the squares commute.

Example:

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d] & Y \arrow[d] \\ 0 \arrow[r] & F \end{tikzcd}$$

homotopy commutative

If you believe in homotopy cokernels

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d] & Y \arrow[d] \arrow[ddr, bend left] & \\ 0 \arrow[r] \arrow[drr, bend right] & C(f) \arrow[dr, dashed] & \\ & & F \end{tikzcd}$$

TR3

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d] & Y \arrow[r] \arrow[d] & C(f) \arrow[r] \arrow[d, dashed, "\exists !"] & X[1] \arrow[d] \\ 0 \arrow[r] & F \arrow[r, "id"] & F \arrow[r] & 0[1]=0 \end{tikzcd}$$

~~the issue is~~ So $C(f)$ plays the role of cokernel. But the issue is that ~~we~~ this arrow is not unique and it is because we forgot htpy which establishes commutativity of square.

~~Interpretation of~~

(TR4)
$$\begin{tikzcd} X \arrow[r, "x"] & Y \arrow[dl] \\ A \arrow[u, dashed] & \end{tikzcd} , \quad \begin{tikzcd} Y \arrow[r, "y"] & Z \arrow[dl] \\ C \arrow[u, dashed] & \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} X \arrow[r, "y \circ x"] & Z \arrow[d] \\ & B \arrow[ul, dashed] \end{tikzcd}$$

$\exists$ triangle
$$\begin{tikzcd} A \arrow[r] & B \arrow[dl] \\ C \arrow[u, dashed] & \end{tikzcd}$$
satisfying something.

Homotopical thinking:

$$\begin{tikzcd} X \arrow[r] & Y \arrow[dl] \\ A \arrow[u, dashed] & \end{tikzcd} \iff \begin{tikzcd} X \arrow[r] \arrow[d] & Y \arrow[d] \arrow[dl, phantom, "\Swarrow"] \\ 0 \arrow[r] & A \arrow[ul, phantom, "\lrcorner", very near start] \end{tikzcd}$$

So these three triangles give us the following push-out squares:

$$\begin{tikzcd} X \arrow[r] \arrow[d] & Y \arrow[d] \arrow[dl, phantom, "\Swarrow"] \\ 0 \arrow[r] & A \end{tikzcd} \qquad \begin{tikzcd} Y \arrow[r] \arrow[d] & Z \arrow[d] \arrow[dl, phantom, "\Swarrow"] \\ 0 \arrow[r] & C \end{tikzcd}$$

$$\begin{tikzcd} X \arrow[r] \arrow[d] & Z \arrow[d] \arrow[dl, phantom, "\Swarrow \gamma"] \\ 0 \arrow[r] & B \end{tikzcd}$$

(1)

$$\begin{tikzcd} X \arrow[r] \arrow[d] \arrow[dr, phantom, "\swarrow"] & Y \arrow[r] \arrow[d] & Z \arrow[d] \\ 0 \arrow[r] & A \arrow[r] & B \end{tikzcd} \quad \Longrightarrow \quad \begin{tikzcd} Y \arrow[r] \arrow[d] \arrow[dr, phantom, "\swarrow"] & Z \arrow[d] \\ A \arrow[r] & B \end{tikzcd}$$

An easy lemma about push-outs

(2)

$$\begin{tikzcd} Y \arrow[r] \arrow[d] \arrow[dr, phantom, "\swarrow"] & Z \arrow[d] \\ A \arrow[r] \arrow[d] & B \arrow[d] \\ 0 \arrow[r] & C \end{tikzcd} \quad \Longrightarrow \quad \begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[d] \\ 0 \arrow[r] & C \end{tikzcd}$$

So (TR3) and (TR4) are somehow properties about ~~mapping cones~~ homotopy cokernel.

<u>Remark</u> (TR3) ~~[illegible]~~ can be derived by other axioms. (Peter May's observation)

Bridgeland:

Specify a heart for D (recovers monos by [1])

Stab cond. for Abelian Categories

Stab condition for Δ'd categories

Declare certain objects to be semi-stable

Defn ① A stability condition for D triangulated is a pair $(Z, \heartsuit)$ where $\heartsuit$ is a heart (of a bdd t-structure) $Z: \mathrm{ob}\,\heartsuit \to \mathbb{C}$ is a B.S.C for $\heartsuit$.

Defn ② A stab cond. for D is a pair (Z, P) where

- $Z: K_0(D) \to \mathbb{C}$ gp homo.
- P is a "slicing" $P = \{P(\phi)\}_{\phi \in \mathbb{R}}$ for D. (slicing is defined below)
- $\forall\ A \in P(\phi)\ \exists\ m > 0$ s.t. $Z(A) = m e^{i\pi\phi}$

Rmk How does one determine the semi-stable objects using def^n ①?

We know what it means for $A \in \heartsuit$ to be semi-stable.

Defn $\forall\ \phi \in (0,1]$ $\quad P(\phi) =$ full sub-category of $\heartsuit$ spanned by s.s. obj. of phase ϕ

then $\forall\ \phi' = \phi + N \quad N \in \mathbb{Z}$
$\phi \in (0,1]$

$P(\phi') = P(\phi)[N]$

Prop^n The collection $\{P(\phi)\}_{\phi\in\mathbb{R}}$ satisfies the following:

(sl 0) Each $P(\phi)$ is closed under $\oplus$ and is full in D.

(sl 1) $\forall\, \phi\in\mathbb{R}$ $P(\phi+1)=P(\phi)[1]$

(sl 2) $\forall\, \phi_1>\phi_2$, $A_i\in P(\phi_i)$
$$\hom_D(A_1,A_2)=0$$

(sl 3) $\forall\, E\in D$ $\exists$ sequence of triangles:

$$\begin{tikzcd}
0=E_0 \arrow[r] & E_1 \arrow[r] \arrow[d] & E_2 \arrow[r, dashed] \arrow[dl] & \cdots \arrow[r] & E_n=E \arrow[d] \\
 & A_1 \arrow[ul, dashed] & A_2 \arrow[u, dashed] & & A_n \arrow[ul, dashed]
\end{tikzcd}$$

s.t. $A_i\in P(\phi_i)$ and $\phi_i>\phi_{i+1}$ $\forall\, i$.

Defn Any collection $P=\{P(\phi)\}_{\phi\in\mathbb{R}}$ is called *slicing* of D if it satisfies (sl 0)$\sim$(sl 3).

Rmk This defn is convenient for defining/analyzing topology of the set of stab cond.

Pf (sl 0) Full by definition. Also, if $E, E'\in$ ob$\heartsuit$ with $\phi(E)=\phi(E')$ then $E\oplus E'$ is semi-stable and $\phi(E\oplus E')=\phi(E)$

(sl 1) by def^n

(sl 2) If $\phi_1,\phi_2\in(N, N+1]$ this follows from abelian case.

$$\hom_D(A_1, A_2) \cong \hom_D(A_1[-N], A_2[-N]) = 0$$

(See 1st day of class)

Otherwise, it follows from def$^{\underline{n}}$ of $\heartsuit$.

(Since $\hom(\heartsuit[k_1], \heartsuit[k_2]) = 0$)

if $k_1 > k_2$

(Sl 3) By def$^{\underline{n}}$ of $\heartsuit$ we have a filt. of any $E \in D$.

$$\begin{tikzcd}
0 = E_0 \arrow[r] & E_1 \arrow[r] \arrow[d] & E_2 \arrow[r] \arrow[d] & \cdots \arrow[r] & E_n = E \arrow[d] \\
& B_1 \arrow[ul, dashed] & B_2 \arrow[ul, dashed] & & B_n \arrow[ul, dashed]
\end{tikzcd}$$

where $B_i \in \heartsuit[k_i]$ and $k_i > k_{i+1}$ $\forall i$. Since Z is a BSC for $\heartsuit$, $\exists$HN filtration for each $B_i[-k_i] \in \heartsuit$.

<u>Ex</u>:

$$\begin{tikzcd}
E_i \arrow[r] \arrow[d] & \cdots & \widetilde{A}_{i+1,j} \arrow[r] \arrow[dd] & E_{i+1} \arrow[d] \\
0 = A_{i+1,0} \arrow[r] & \arrow[r] & \cdots \arrow[r] & B_{i+1} \arrow[d] \\
& A_{i+1,1} & \cdots & A_{i+1,N}
\end{tikzcd}$$

<u>Rmk</u>: Strictly speaking we used the fact that the pull-back squares are push-out and for that we need for example to work with stable ∞-Cat's. In general, we need to work further to establish the result for Δ'd cat's.

Defn Given a slicing $\mathcal{P}$ of $\mathcal{D}$ and two real numbers $a \leq b \in \mathbb{R}$. Let $\mathcal{P}((a,b]) \subseteq \mathcal{D}$ be the full sub-cat. spanned by all $E \in ob(\mathcal{D})$ where H-N factors have phase in $(a,b]$.

Rmk Uniqueness of the phases of H-N filtrations depends on $\mathcal{P}$ being compatible w/ some

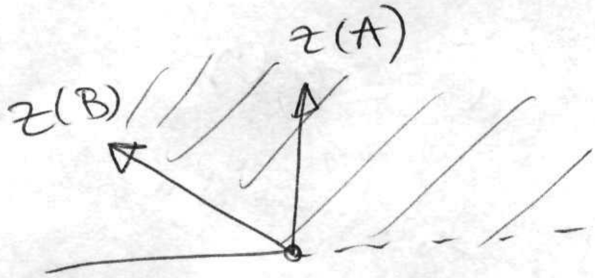
$$Z : K_0(\mathcal{D}) \longrightarrow \mathbb{C}.$$

Rmk We can define $\mathcal{P}(I)$ for any interval.

~~Prop~~

⚠ $\mathcal{P}(I) \neq$ Full sub cat. spanned by $E \in ob(\mathcal{D})$ w/ $\phi(E) \in I$

Ex:



$I = (0, 1]$

$A \oplus A[-1] \oplus B$ has phase in I but it is not in $\mathcal{P}(I)$.

$\underline{\text{Prop}^n}$ Let $(Z, \mathcal{P})$ be a B.S.C for D. $\left(\begin{array}{l} Z : K_0(D) \to \mathbb{C} \\ \{\mathcal{P}(\phi)\}_{\phi \in \mathbb{R}} \end{array} \right)$

Then for each $\phi \in \mathbb{R}$, $\mathcal{P}((\phi, \phi+1])$ is a heart for D.

- $Z(A) = m e^{i\pi\phi}$ $m > 0$, $A \in \mathcal{P}(\phi)$

$\underline{\underline{\text{Pf}}}$ (H1) Let $E \in \mathcal{P}((\phi, \phi+1])$ and $E' \in \mathcal{P}((\phi+N, \phi+N+1])$ w/ $N > 0$

We need to show $\hom_D(E', E) = 0$.

(Case: E' semi-stable) Let $f : E' \to E$.

Consider H-N filt. of E:

Observation: Given H-N filt. of $E \in \mathcal{P}((\phi, \phi+1])$

$$\begin{array}{ccc} & & E' \\ & \swarrow^{f_{n-1}} & \downarrow \\ \cdots \to E_{n-1} & \to & E_n = E \\ & \nwarrow & \downarrow \\ & & A_n \end{array}$$

$\phi(A_1) > \phi(E) > \phi(A_n)$

As such $\phi(A_n) < \phi(E) < \phi(E')$.

By (sl 2) the map $g_n : E' \to A_n$ must be zero. By induction assume we have a map $f_{k-1} : E' \to E_{k-1}$.

we have a sequence of maps

$$\begin{tikzcd} & E' \arrow[dl, "f_0"'] \arrow[d] \\ 0 = E_0 \arrow[r] & E \end{tikzcd}$$

hence $E' \to E_n = 0$.

(Case II: E' is not semi-stable)

Examine H-N filt for E':

$$0 = E'_0 \to E'_1 \to \cdots \to E'_m = E'$$

then $f|_{E_1} = 0 \Rightarrow f$ induces a map:

$$g_2 : E'_2 / E' \to E$$

As before, $g_2 = 0$. So we can assume by induction that $f|_{E'_k} = 0 \Rightarrow f|_{E} = f = 0$

$$\begin{tikzcd} E'_k \arrow[r] \arrow[d] \arrow[ddrr, bend right=60, "f|_{E'_k}"'] & E'_{k+1} \arrow[d] \arrow[ddr, bend left, "f|_{E'_{k+1}}"] & \\ 0 \arrow[r] & A_{k+1} \arrow[dr, dashed, "0"] & \\ & & E \end{tikzcd}$$

Pf of H-2:

Given a H-N filt (by 5(3))

$$\begin{tikzcd} E_0 \arrow[r] & E_1 \arrow[r] & \circ \arrow[r, dashed] \arrow[d] & \circ \arrow[r] \arrow[d] & E_n = E \\ & & A_1 & A_k & \end{tikzcd}$$

just group the E_i together where H-N factors lie in $(\phi + N, \phi + N + 1]$.