TR3 & TR4

Given a commutative diagram
$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "g"'] & Y \arrow[d] \arrow[dl, phantom, "\Swarrow"] \\ X' \arrow[r, "f'"'] & Y' \end{tikzcd},$$
we obtain a diagram
$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d] & Y \arrow[r] \arrow[d] & C(f) \arrow[r] & X[1] \arrow[d, "g[1]"] \\ X' \arrow[r, "f'"'] & Y' \arrow[r] & C(f') \arrow[r] & X'[1] \end{tikzcd}$$

(TR3) states that there exists a map $C(f) \to C(f')$ making the diagram commute.

If one believes in homotopy co-kernels
$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & Y \arrow[d] \\ 0 \arrow[r] & C(f) \end{tikzcd},$$
the map $C(f) \longrightarrow C(f')$ should be unique up to a contractible choice. It is not unique in the derived category because in the derived category we do not remember the homotopies that make the original diagram commute. In the world of stable ∞-categories, we simply write
$$\begin{tikzcd} & X \arrow[rr] \arrow[dl] \arrow[dd] & & y \arrow[dl] \arrow[dd] \\ X' \arrow[rr] \arrow[dd] & & y' \arrow[dd] & \\ & 0 \arrow[rr] \arrow[dl] & & C(f) \arrow[dl, "\exists!"] \\ 0 \arrow[rr] & & C(f') & \end{tikzcd}$$

(TR4) states that, given triangles

$$
\begin{tikzcd}[column sep=small]
X \arrow[rr, "x"] & & Y \arrow[dl] \\
& A \arrow[ul, dashed] &
\end{tikzcd},
\quad
\begin{tikzcd}[column sep=small]
Y \arrow[rr, "y"] & & Z \arrow[dl] \\
& C \arrow[ul, dashed] &
\end{tikzcd},
\quad \& \quad
\begin{tikzcd}[column sep=small]
X \arrow[rr, "y \circ x"] & & Z \arrow[dl] \\
& B \arrow[ul, dashed] &
\end{tikzcd},
$$

there exists a triangle
$$
\begin{tikzcd}[column sep=small]
A \arrow[rr] & & B \arrow[dl] \\
& C \arrow[ul, dashed] &
\end{tikzcd}
$$
satisfying a few conditions I will not spell out.

This axiom will become clearer with some <u>homotopical thinking</u>. We can rewrite the three triangles as homotopy pushout squares.

$$
\begin{tikzcd}
X \arrow[r] \arrow[d] \arrow[dr, phantom, very near start, "\ulcorner"] & Y \arrow[d] \arrow[dl, Rightarrow] \\
0 \arrow[r] & A
\end{tikzcd},
\quad
\begin{tikzcd}
Y \arrow[r] \arrow[d] \arrow[dr, phantom, very near start, "\ulcorner"] & Z \arrow[d] \arrow[dl, Rightarrow] \\
0 \arrow[r] & C
\end{tikzcd},
\quad \text{and} \quad
\begin{tikzcd}
X \arrow[r] \arrow[d] \arrow[dr, phantom, very near start, "\ulcorner"] & Z \arrow[d] \arrow[dl, Rightarrow] \\
0 \arrow[r] & B
\end{tikzcd}.
$$

In the diagram

$$
\begin{tikzcd}
X \arrow[r] \arrow[d] & Y \arrow[r] \arrow[d] & Z \arrow[d] \\
0 \arrow[r] & A \arrow[r] & B
\end{tikzcd}
$$

$$\underbrace{\underbrace{\hphantom{0 \longrightarrow A}}_{\text{pushout}}\hphantom{\longrightarrow B}}_{\text{pushout}}$$

the two marked pushout squares imply by abstract nonsense that
$$
\begin{tikzcd}
Y \arrow[r] \arrow[d] & Z \arrow[d] \\
A \arrow[r] & B
\end{tikzcd}
$$
is a pushout as well.

Examination of the diagram

pushout

$$\begin{tikzcd} Y \arrow[r] \arrow[d] & Z \arrow[d] \\ A \arrow[r] \arrow[d] & B \arrow[d] \\ 0 \arrow[r] & C \end{tikzcd}$$

pushout

then implies that $\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[d] \\ 0 \arrow[r] & C \end{tikzcd}$ is a pushout square as well.

<u>Stability Conditions</u>

Recall the two definitions we gave for a BSC on a Δ'd category D:

<u>Def 1</u>: A stability condition is a pair $(Z, \heartsuit)$ where $\heartsuit \subset D$ is a heart of a bounded t-structure & $Z: \mathrm{ob}\, D \to \mathbb{C}$ is a BSC for $\heartsuit$.

<u>Def 2</u>: A stability condition for D is a pair (Z, P) where

- $Z: K_0(D) \longrightarrow \mathbb{C}$ is a group homomorphism
- P is a slicing for D, $P = \{P(\phi)\}_{\phi \in \mathbb{R}}$
- $\forall A \in P(\phi)$, $\exists m > 0$ s.t.
$$Z(A) = m e^{i\pi\phi}.$$

<u>Remark</u>: Using definition 1, one determines the semi-stable objects by recalling what it means for $A \in \heartsuit$ to be semi-stable.

Let us recall the definition of a slicing by axiomatizing the properties of the slicing which naturally occurs in the situation of definition 1.

<u>Def</u>: $\forall \phi \in [0,1]$, $P(\phi) =$ full subcategory of $\heartsuit$ spanned by semi-stable objects of phase ϕ.

For $\phi' = \phi + N$, $N \in \mathbb{Z}$, we set $P(\phi') = P(\phi)[N]$

<u>Prop</u>: The collection $\{P(\phi)\}_{\phi \in \mathbb{R}}$ satisfies

(Sl 0) Each $P(\phi)$ is closed under $\oplus$ and is full.

(Sl 1) For all $\phi \in \mathbb{R}$, $P(\phi + 1) = P(\phi)[1]$

(Sl 2) $\forall \phi_1 > \phi_2$, $A_i \in P(\phi_i)$,

$$\hom_D(A_1, A_2) = 0.$$

(Sl 3) $\forall E \in D$, $\exists$ sequence of triangles

$$\begin{tikzcd}
0 = E_0 \arrow[r] & E_1 \arrow[r] \arrow[d] & E_2 \arrow[r] \arrow[d] & \cdots \arrow[r] & E_n = E \arrow[d] \\
& A_1 \arrow[ul, dashed] & A_2 \arrow[ul, dashed] & & A_n \arrow[ul, dashed]
\end{tikzcd}$$

such that $A_i \in P(\phi_i)$ and $\phi_i > \phi_{i-1}$ for all i.

<u>Def</u>: Any collection $P = \{P(\phi)\}_{\phi \in \mathbb{R}}$ is called a <u>slicing</u> of the triangulated category P if (Sl 0) – (Sl 3).

<u>Remark</u>: This definition is convenient for defining and analyzing the topology of the set of stability conditions.

Def: Given a slicing P of $\mathcal{D}$ and two real numbers $a \le b \in \mathbb{R}$, let

$$P((a,b]) \subset \mathcal{D}$$

denote the full subcategory spanned by all $E \in ob\,\mathcal{D}$ where H-N factors have phase in $(a,b]$.

Remark: Uniqueness of the phases in the H-N filtration depends on P being compatible with some $Z: K_0(\mathcal{D}) \longrightarrow \mathbb{C}$.

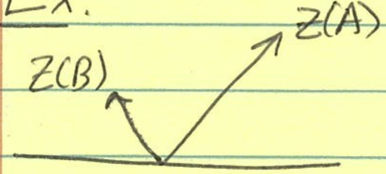
Remark: We can define $P(I)$ for any sort of interval I, not merely the half-open and half-closed ones.

Prop: Let (Z, P) be a BSC $\left(\begin{array}{l} Z: K_0(\mathcal{D}) \rightarrow \mathbb{C} \\ \{P(\phi)\}_{\phi \in \mathbb{R}} \\ Z(A) = m e^{i\pi\phi}, \quad m>0 \\ \text{for } A \in P(\phi) \end{array} \right)$

Then $\forall \phi \in \mathbb{R}$, $P((\phi, \phi+1])$ is a heart (of a bounded t-structure) for $\mathcal{D}$.

Warning: $P(I) \ne$ Full subcategory spanned by $E \in ob\,\mathcal{D}$ with $\phi(E) \in I$.

Ex:



$I = (0, 1]$

$A \oplus A[-1] \oplus B$ has phase in I but is NOT in $P(I)$.

Pf: (H1) Let $E \in \mathcal{P}(\phi, \phi+1)$,

$E' \in \mathcal{P}(\phi+N, \phi+N+1)$,

$N > 0$.

We need to show that $\hom_{\mathcal{D}}(E, E') = 0$.

If E' is semi-stable, let

$$f : E' \longrightarrow E$$

and examine the H-N filtration of E

$$\begin{tikzcd}
 & & E' \arrow[d] \\
\cdots \arrow[r] & E_{n-1} \arrow[r] & E_n = E \arrow[dl] \\
 & A_n \arrow[u, dashed] &
\end{tikzcd}$$

Observation: Given a H-N filtration of $E \in \mathcal{P}((\phi, \phi+1])$,

$\phi(A_1) > \phi(E) > \phi(A_n)$.

As such, $\phi(A_n) < \phi(E) < \phi(E')$. By (Sl 1), the map $E' \longrightarrow A_n$ must be 0, and so we have a map $E' \longrightarrow E_{n-1}$. We can induct on k, assuming we have a map $f_{k-1} : E' \longrightarrow E_{k-1}$.