What should a stability condition on a triangulated D be?

(a) We should have a stability function

$$\mathrm{ob}\, D \xrightarrow{Z} \mathbb{C}$$

respecting exact triangles.
(distinguished)

(b) Some <u>ordering</u> on objects.

(c) A notion of semistable objects.

(d) A Harder-Narasimhan property.

I want to now convince you of two things:

(b'): It doesn't make sense to try and order <u>all</u> objects of $\mathcal{D}$.
(In contrast to Abelian case.)

We should only expect an ordering of <u>semistable</u> objects.

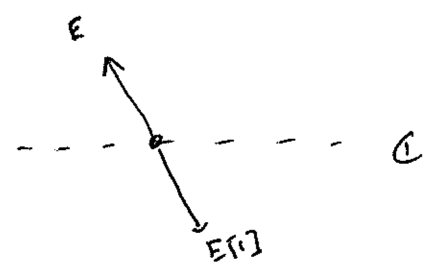
(c'): Being semistable can't be a ~~property~~ of an object coming simply from Z; we need some extra <u>data</u>.

Like a choice of heart.

Or a downright declaration.

<u>Regarding (b')</u>: What was our ordering in the abelian case?

A <u>phase</u> $\phi(E) \text{ "=" } \mathrm{Arg}(Z(E))$.



But non-zero objects can have zero central charge!

$$Z(E \oplus E[1]) = Z(E) + Z(E[1])$$
$$= 0.$$

On the other hand, if an object is "indecomposable" enough, we can expect it to have non-zero phase, and hence an ordering.

What's a kind of object that seems "indecomposable enough"?

The semistable ones.

<u>Regarding (c')</u>:

- If we know that

$$\mathcal{D} = D^b(\heartsuit)$$

for some abelian category $\heartsuit$, then there's a natural choice of "semistability". (Natural, if not naive.) Try

$$\begin{gathered} E \text{ is semistable} \\ \Updownarrow \\ E \simeq E'[n] \text{ for some } \\ E' \in \mathrm{ob}\,\heartsuit \text{ semistable}. \end{gathered}$$

But this depends on <u>knowing</u> a heart!

- On the other hand, what if we had no heart for $\mathcal{D}$?

We're shit out of luck, since (a) alone can't determine (b) and (c)

a fxn Z

an ordering

ss objects

So we should just <u>declare</u> some objects to be semistable.

This leads to the notion of <u>slicing</u>.

- Finally, "indecomposable" is a really hard concept to formalize w/out extra data. For instance, there's no good notion of a "subobject" since any $E' \longrightarrow E$ can be extended to an exact triangle.

Defn A Bridgeland stability condition on $\mathcal{D}$ is a pair

$$(\heartsuit, Z)$$

where $\heartsuit \subset \mathcal{D}$ is a heart, and Z is a stability condition on $\heartsuit$.

Ex $\mathcal{D} = D^b \mathrm{Rep}^{f.d.} Q$, $\heartsuit = \mathrm{Rep}^{f.d.} Q$

$$Z : \mathrm{ob}\, \mathrm{Rep}^{f.d.} Q \longrightarrow \mathbb{C}$$
$$V \longmapsto \sum \dim V_i \cdot z_i .$$

Ex $D = D^b \mathrm{Coh}\, X$, $\heartsuit = \mathrm{Coh} X$, X a curve.

$$Z : \mathrm{ob}\, \mathrm{Coh} X \longrightarrow \mathbb{C}$$
$$E \longmapsto i \,\mathrm{rk}\, E - \deg E .$$

What can we do given a pair $(\heartsuit, Z)$?

- $\forall\ \phi \in (0,1]$, let

  $P(\phi) :=$ full subcategory of $E \in \mathrm{ob}\,\heartsuit$ s.t.

  - E semistable
  - $\phi(E) = \phi$.

- $\forall\ \phi \in \mathbb{R}$, $\phi = \phi_0 + N$ (w/ $\phi_0 \in (0,1]$) let

  $P(\phi) := P(\phi_0)[N]$.

<u>Rmk</u> Each $P(\phi)$ is closed under $\oplus$ since

E, E' semistable of phase ϕ

$\Longrightarrow$ $E \oplus E'$ semistable, of phase ϕ.

Prop'n The collection of subcategories

$$\{P(\phi)\}_{\phi\in\mathbb{R}}$$

satisfies the following:

(SI 1) $\forall\, \phi\in\mathbb{R}$,

$$P(\phi+1) = P(\phi)[1]$$

(SI 2) $\forall\ \phi_1 > \phi_2$, $A_i \in P(\phi_i)$,

$$\hom_D(A_1, A_2) = 0$$

(SI 3) $\forall\ E\in D$, $\exists$ triangles

$$\begin{tikzcd}
0 = E_0 \arrow[r] & E_1 \arrow[r] \arrow[d] & E_2 \arrow[r] \arrow[d] & \cdots \arrow[r] & E_{n-1} \arrow[r] & E_n = E \arrow[d] \\
& A_1 \arrow[ul, dashed] & A_2 \arrow[ul, dashed] & & & A_n \arrow[ull, dashed]
\end{tikzcd}$$

s.t. $A_i \in P(\phi_i)$, $\phi_i > \phi_{i+1}$ $\forall\, i$.

Pf.

(Sl 1) by definition

(Sl 2) (a) If $\phi_1, \phi_2 \in [N, N+1]$,

this follows because

$$\hom_D(A_1, A_2) \cong \hom_D(A_1[-N], A_2[-N])$$

$$= 0$$

[1] is an autoequivalence

by proposition from our first class.

(b) Otherwise, it follows by definition of hearts

(a) (b)

(Sl 3) Let

$$\begin{tikzcd} E_0' \arrow[r] & E_1' \arrow[r] \arrow[dl] & \cdots \arrow[r] & E_{N'}' = E \arrow[dl] \\ A_1' \arrow[u, dashed] & & A_{N'}' \arrow[u, dashed] & \end{tikzcd}$$

be the decomposition of E into objects of $\heartsuit[i']$.

Use H-N property of Z to decompose each $A_{i'}'$ into semistable A_i.

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$$
\begin{tikzcd}
0 = E_0 \arrow[r] & E_1 \arrow[r] \arrow[d] & E_2 \arrow[r] \arrow[d] & E_3 \arrow[r] \arrow[d] & E_4 \arrow[d] \\
 & 0 \arrow[r] & A_2 \arrow[r] \arrow[d] & E_3/E_2 \arrow[r] \arrow[d] & E_4/E_2 \arrow[d] \\
 & & 0 \arrow[r] & A_3 \arrow[r] \arrow[d] & E_4/E_3 \arrow[d] \\
 & & & 0 \arrow[r] & A_4 \arrow[d] \\
 & & & & 0
\end{tikzcd}
$$

all squares pushouts. $\boxed{A}$

Use H-N!

More details on (Sl 3)

$$
\begin{tikzcd}
0 = B^3_0 \arrow[r] & B^3_1 \arrow[r] \arrow[d] & B^3_2 \arrow[r] \arrow[d] & \cdots \arrow[r] & B^3_{N_3} = A_3 \\
 & 0 \arrow[r] & C^3_2 \arrow[d] & & \\
 & & 0 \arrow[r] & C^3_3 \arrow[d] & \\
 & & & 0 \cdots & \\
 & & & & C^3_{N_3} \arrow[d] \\
 & & & & 0
\end{tikzcd}
$$

Rmk That these pullback squares $\tilde{B} \dots$ are also pushout squares is a property of stable ∞-categories. In the setting of triangulated cats, you have to manipulate rotations of triangles.

Insert into $\boxed{A}$

$$
\begin{tikzcd}
\cdots E_2 \arrow[r] \arrow[d] & \tilde{B} \arrow[r] \arrow[dd] \arrow[ddr, phantom, "\text{p.b.}"] & \tilde{B} \arrow[r] \arrow[dd] \arrow[ddr, phantom, "\text{p.b.}"] & \tilde{B} \arrow[r] \arrow[dd] \arrow[ddr, phantom, "\text{p.b.}"] & \tilde{B} \arrow[r] \arrow[dd] \arrow[ddr, phantom, "\text{p.b.}"] & E_3 \cdots \arrow[dd, dashed] \\
A_2 \arrow[d] & & & & & \\
0 \arrow[r] & B_1^3 \arrow[r] \arrow[d] & B_2^3 \arrow[r] & B_3^3 \arrow[r] & \cdots \arrow[r] & B^3_{N_3} \arrow[d, dashed] \\
 & C_2^3 \arrow[r] & & & & \cdots
\end{tikzcd}
$$

This gives the HN filtration for E.

__Defn__ Any collection of full, additive subcategories

$$\{ P(\phi) \subset D \}_{\phi \in \mathbb{R}}$$

is called a __slicing__ of D if it satisfies

$$(Sl1) \sim (Sl3).$$

__Rmk.__ This will be convenient for defining a topology on $\mathrm{Stab}(D)$.

__Defn__ Given a slicing P of D, and two real numbers $a \leq b \in \mathbb{R}$, let

$$P((a, b])$$

be the full category spanned by all $E \in \mathrm{ob}\, D$ whose H-N factors have phase in $(a, b]$.

Rmk This $P(I)$ can be defined for any interval I — closed, open, whatever.

Rmk $P(I)$ is equivalent to the extension closure of

$$\bigcup_{\phi \in I} P(\phi).$$

Prop. Let (Z, P) be a Bridgeland Stability Condition.

Then $\forall\ \phi \in \mathbb{R}$,

$$P(\phi, \phi+1]$$

is a heart for (a bdd t-structure on) $\mathcal{D}$.

Pf.

(H1) Let $E \in P((\phi, \phi+1])$

$E' \in P((\phi+N, \phi+N+1])$, $N > 0$. Note $P((\phi+N, \phi+N+1]) = P((\phi, \phi+1])[N]$.

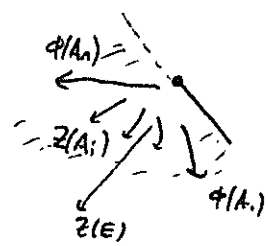
- If E' semistable, NTS $\hom_D(E', E) = 0$.

Take H-N filt. of E.

$$\begin{tikzcd}
 & & & E' \arrow[d, "f_n"] \\
0 = E_0 \arrow[r] & \cdots \arrow[r] & E_{n-1} \arrow[r] & E_n = E \arrow[dl, "p_n"] \\
 & & A_n \arrow[u, dashed] &
\end{tikzcd}$$

Let $g_n : E' \to A_n$ be the composition $p_n \circ f_n$.

- Since $\phi(A_i) \in$ the unit interval $(\phi, \phi+1]$ $\forall\, i$,



we know $\phi(A_1) > \phi(E) > \phi(A_n)$.

So $\phi(A_n) < \phi(E') \to g_n = 0$.

This induces ~~the~~ map

$$\begin{tikzcd}
 & & E' \arrow[dl, "f_{n-1}"'] \\
E_n \arrow[r] & E_{n-1} \arrow[r] \arrow[d, "p_{n-1}"] & E_n \\
 & A_{n-1} &
\end{tikzcd}$$

- Again, $\phi(A_{n-1}) < \phi(E_{n-1})$, while $\phi(E_{n-1}) \in (\phi, \phi+1]$.

So $g_{n-1} = p_{n-1} \circ f_{n-1} = 0$. This induces a map

$$\begin{tikzcd}
 & & E' \arrow[dll, "f_{n-2}"'] \arrow[dl] \arrow[d] \\
E_{n-2} \arrow[r] & E_{n-1} \arrow[r] & E
\end{tikzcd}$$

- By induction, we obtain a map $E' \xrightarrow{f_0} E_0 = E$ factoring f_n. So $f_n = 0$.

Pf of (H1)' cont'd.

- If E' NOT semistable, consider H-N filtration of E'!

$$\begin{tikzcd} 0 = E_0' \arrow[r] & E_1' \arrow[r] & \cdots \arrow[r] & E_m' = E' \arrow[d, "f"] \\ & & & E. \end{tikzcd}$$

We begin w/ E_1' (in contrast to E_n).

- E_1' is semistable by defn of H-N filt, so by previous argument,

$$f|_{E_1'} = 0.$$

- $\exists$ induced map $\quad g_2 : \underset{\displaystyle A_2'}{\overset{\shortparallel}{E_2'/E_1'}} \longrightarrow E.$

$$\begin{tikzcd} E_1' \arrow[r] \arrow[d] \arrow[dd, bend right, "f|_{E_1'}"'] & E_2' \arrow[ddr, "f|_{E_2'}"] & \\ 0 \arrow[d, phantom, "\Swarrow"] \arrow[drr] & & \\ {} & & E \end{tikzcd}$$

But A_2' semistable, so by previous argument, $g_2 = 0$.

so

$\exists$

$$\begin{tikzcd} E_1' \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{p.o.}"] & E_2' \arrow[d] \arrow[ddr, bend left] & \\ 0 \arrow[r] \arrow[drr, bend right] & A_2' \arrow[dr, "\exists"] & \\ & & E \end{tikzcd}$$

- $f|_{E_{k-1}'} = 0 \Rightarrow$ induced map
$$g_k : \underset{\displaystyle A_k'}{\overset{\shortparallel}{E_k'/E_{k-1}'}} \longrightarrow E$$

But A_k' semistable, so $g_k = 0$.

- By induction, $f|_{E_m'} = f = 0$.

Pf of (H2)

Given H-N filtration of (Sl 3),

$$0 = E_0 \to E_1 \to E_2 \to \cdots \to E_n = E$$

$\forall\ N \in \mathbb{Z}$, let i_N be the number such that

$$i_N = \max_j \left\{ j \,\middle|\, \phi\left(E_j/E_{j-1}\right) \in (\phi+N, \phi+N+1] \right\}.$$

Let $N_1, \ldots, N_k$ be those integers for which

$i_{N_j} \neq -\infty$. (max of empty set $= -\infty$).

Then $0 = E_0 \to E_{N_1} \to E_{N_2} \to \cdots \to E_{N_k} = E$

is a filtration as in (H2).

(Because $E_{N_j}/E_{N_{j-1}}$ is an extension of semistables w/ phase in $(\phi+N_j, \phi+N_j+1)$, hence is an object of $\mathcal{P}(\phi+N_j, \phi+N_j+1) = \mathcal{P}(\phi, \phi+1][N_j]$.)

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