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Equivalence of the following two definitions: **Let** D be a triangulated category.

Def A Bridgeland stability condition on D is $(Z, \overset{\text{slicing}}{P})$ or $(Z, \overset{\text{heart}}{\heartsuit})$.

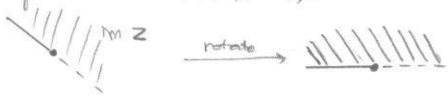
Last time:

Prop Given (Z,P), then $\forall\ \phi_0 \in \mathbb{R}$, $P((\phi_0, \phi_0+1])$ is the heart of a bounded t-structure on D.

$$\begin{tikzcd} \{(Z,P)\} \arrow[r, leftrightarrow] \arrow[r, bend left, "\heartsuit = P((0,1])"] & \{(Z,\heartsuit)\} \arrow[l, bend left, "{P(\phi+n) := P(\phi)[n],\ \phi\in(0,1]}"] \end{tikzcd}$$

Prop $\forall\ \phi_0 \in \mathbb{R}$, Z defines a Bridgeland stability condition on $P((\phi_0, \phi_0+1])$.

Proof Really, we rotate Z by $e^{-i\pi\phi_0}$



(0) satisfied.

(1) Let $0 \neq E \in P((\phi_0, \phi_0+1])$. E has Harder-Narasimhan filtration by $A_i \in P(\phi_i)$ by (Sl 2). $Z(A_i)$ are all in upper half plane, so $\sum Z(A_i) \neq 0$.

(2) obvious $\because$ SES in $\heartsuit$ is an exact triangle in D all of whose objects are in $\heartsuit$.

(3) obvious by defn of slicing.

Prop If $A \in ob\, P(\phi)$, $\phi \in (\phi_0, \phi_0+1]$, then A is a Z-semistable object in $P((\phi_0, \phi_0+1])$.

Proof WLOG $\phi_0 = 0$ Let $\phi(\overset{\overset{(0,1]}{\in}}{E}) > \phi(A)$, $E = P((0,1])$ and consider any map $f: E \to A$. By H-N filtration on E,

$E_1 \to E_2 \to \cdots \to E_n = E$, so $\phi(E_1) > \phi(E)$.

So composition $E_1 \hookrightarrow E \xrightarrow{f} A$ is zero. (since E_1, A semistable, $\phi(E_1) > \phi(A)$.) So f must have kernel.

On the other hand, let $A \in P((0,1])$ be Z-semistable (in the sense of abelian categories). If A is not its own Harder-Narasimhan filtration, $\exists\ E_1 \hookrightarrow A$ with $\phi(E_1) > \phi(A)$ $\Longrightarrow\!\Longleftarrow$

Topology on Stab (D).

Def Let P be a slicing for D. $\forall E$,
$$\begin{tikzcd} 0 \arrow[r] & E_1 \arrow[r] \arrow[d, equal] & \cdots \arrow[r] & E_n \arrow[r] \arrow[d] & E \\ & A_1 & & A_n & \end{tikzcd}$$
let $\phi^+_P(E) := \phi_P(A_1)$, $\phi^-_P(E) := \phi_P(A_n)$.

Ex $E \in P([\phi^-_P(E), \phi^+_P(E)])$.

Def Given two slicings P, Q, define $\operatorname{dist}(P,Q) := \sup\limits_{0 \neq E \in ob(D)} \{ |\phi^+_P(E) - \phi^+_Q(E)|, |\phi^-_P(E) - \phi^-_Q(E)| \} \in [0, \infty]$

Prop dist satisfies:

- $\operatorname{dist}(P,Q) = 0 \Rightarrow P = Q$ $\quad (P(\phi) = Q(\phi)\ \forall \phi)$
- $\operatorname{dist}(P,Q) = \operatorname{dist}(Q,P)$
- $\operatorname{dist}(P,Q) + \operatorname{dist}(Q,R) \geq \operatorname{dist}(P,R)$

Rmk dist is called a generalised metric ($\because \infty$ be allowed)

Proof • $\phi^+_Q(E) = \phi^+_P(E) \overset{\text{if } E \in P(\phi)}{=\!=} \phi^-_P(E) = \phi^-_Q(E) \Longrightarrow E \in Q(\phi)$

- Second and third point obvious.

Rmk Two slicings could have $P((0,1]) \cong Q((0,1])$ but still be distinct.

Rmk If $\operatorname{dist}(P,Q) < \varepsilon$, then $P(\phi) \subset Q((\phi-\varepsilon, \phi+\varepsilon))$ and $Q(\phi) \subset P((\phi-\varepsilon, \phi+\varepsilon))$.

<u>Fact</u> (Bridgeland) $\text{dist}(P, Q) = \inf\{\varepsilon \geq 0 \mid Q(\phi) \subset P[\phi - \varepsilon, \phi + \varepsilon]) \quad \forall \phi \in \mathbb{R}\}$

What about Z? i.e, what is the topology on the set of functions $Z : K_0(D) \longrightarrow \mathbb{C}$?

Two approaches: 1) (Bridgeland) Demand (Z, P) be "locally finite", i.e. $\forall \phi \in \mathbb{R}, \exists \varepsilon > 0$ such that $P(\phi - \varepsilon, \phi + \varepsilon))$ is a finite length category

⚠ Warning: $P((\phi - \varepsilon, \phi + \varepsilon))$ may not be abelian, they're "quasiabelian", so you can still discuss sequences of subs and quotients

2) (Kontsevich - Soibelman) Fix a lattice $\Gamma \cong \mathbb{Z}^N$, N finite. (Γ could have torsion, but we ignore this.) Also fix a map $K_0(D) \xrightarrow{cl} \Gamma$.

If we restrict to (Z, P) where Z factors

$$\begin{tikzcd} K_0(D) \arrow[r, "Z"] \arrow[d, "cl"'] & \mathbb{C} \\ \Gamma \arrow[ur, dashed, "Z"'] & \end{tikzcd}$$

then we can induce a topology from $\Gamma^\vee = \hom(\Gamma, \mathbb{C}) \ni Z, W$. ← finite dimensional!

Fix metric on $\Gamma \otimes \mathbb{R} \cong \mathbb{R}^N$ to define norm

$$\| Z \otimes W \| = \sup_{\gamma \in \Gamma} \left\{ \frac{|Z(\gamma) - W(\gamma)|}{\|\gamma\|} \right\}$$

Now we have a metric: given $\sigma = (Z, P)$, $\tau = (W, Q)$.

$\text{dist}(\sigma, \tau) = \max\{\text{dist}(P, Q), \|Z - W\|\}$

<u>Ex</u> Let X be a smooth projective curve $/ \mathbb{C}$, $\mathcal{C} = \text{Coh}(X)$.

There exists a Mukai pairing $\text{ob}\, \mathcal{C} \times \text{ob}\, \mathcal{C} \longrightarrow \mathbb{Z}$

$$(E, F) \longmapsto \sum (-1)^i \text{ext}^i(E, F)$$

<u>Def</u> The numerical Grothendieck group is $K_0(\mathcal{C}) / \{E \mid \langle E, - \rangle \equiv 0\}$.

Take Γ to be this number group.

If X is an elliptic curve, $\Gamma \cong \mathbb{Z}^2 \ni (\deg E, \text{rk}\, E)$

$$\begin{tikzcd} & \Gamma \cong \mathbb{Z}^2 \ni (\deg E, \text{rk}\, E) \\ K_0(\mathcal{C}) \arrow[u] \arrow[ur] & \\ {[E]} \arrow[u, phantom, "\Psi"] \arrow[uur, mapsto] & \end{tikzcd}$$

<u>Ex</u> Let M be a Calabi-Yau 3-fold, $\mathcal{C} = \text{Fuk}(M)$.

$\text{ob}(\mathcal{C})$ are just Lagrangians (exact, monotone)

$L \subset M$, $\dim_{\mathbb{R}} L = \frac{1}{2} \dim_{\mathbb{R}} M$, $\omega|_L \equiv 0$.

Send
$$\begin{tikzcd} {[L]} \arrow[d, mapsto] & K_0(\mathcal{C}) \arrow[d] \\ {[L]} & H_3(M; \mathbb{Z}) \cong \Gamma \end{tikzcd}$$

(maybe $H^3(M; \mathbb{Z})$?)

This should be the charge lattice for some Bridgeland stability condition on $\text{Fuk}(M)$.

middle of Hodge diamond

$$\begin{array}{ccc} & & H^{k,k} \\ \text{middle cohomology of} & \longleftrightarrow & H^{k-1,k-1} \\ \text{Lagrangians} & & \vdots \\ \cap & & H^{0,0} \\ M & & \text{Complex subs} \subset M^\vee \end{array}$$

Fact In this example, one can show

$$\inf\left\{ \frac{\left|\int_L \Omega^{3,0}\right|}{\underset{\text{vol } L}{\underset{\shortparallel}{\|L\|}}},\ L \text{ special} \right\} > 0$$

Def $\sigma = (Z, P)$ is said to satisfy the support property if $\inf\left\{ \frac{|Z(E)|}{\|E\|},\ E \text{ semistable} \right\} > 0$

Rmk Not all Bridgeland stability conditions satisfy (1) local finiteness (2) support property

Ex $\mathcal{C} = \mathrm{Coh}(X)$, X smooth projective curve / $\mathbb{C}$

$$\begin{array}{ccccc} [E] & & K_0(\mathcal{C}) & \longrightarrow & \mathbb{C} \\ \downarrow & & \downarrow & \nearrow & \\ (\deg, \mathrm{rk}) & & \mathbb{Z}^2 & & \end{array}$$

$$[E] \longmapsto \underset{\substack{| \\ \text{fixed element of } \overline{\mathbb{H}} \setminus \mathbb{R}_{>0}}}{Z}\ (\deg E + \alpha \,\mathrm{rk}\, E),\ \alpha \text{ irrational}$$

Exer Show $\mathrm{Coh}(\mathbb{P}^1)$ is not finite length.

Def Fix $cl : K_0(D) \longrightarrow \Gamma$. Let $\mathrm{Stab}(D)$ be the space of $\sigma = (Z, P)$ such that

- Z factors through Γ
- Z satisfies support property

Thm The map $\mathrm{Stab}(D) \longrightarrow \hom(\Gamma, \mathbb{C})$ is a local homeomorphism.

Cor $\mathrm{Stab}(D)$ is a complex manifold.

Lemma $(Z, P) \longrightarrow Z$ is a local injection.

Proof Let $\sigma = (Z, P)$, $\tau = (W, Q)$

We'll show if $d(\sigma, \tau) < \frac{1}{4}$ and $Z = W$, then $P = Q$ $(P(\phi) = Q(\phi)\ \forall \phi)$.

$P(\phi) \subset Q(\phi)$: Let $E \in P(\phi)$. Since it's semistable, $\phi^+_P(E) = \phi(E) = \phi^-_P(E)$.

By definition of dist, $P(\phi) \subset Q((\phi - \frac{1}{4}, \phi + \frac{1}{4})) \subset P((\phi - \frac{1}{2}, \phi + \frac{1}{2})) \subset P((\phi - \frac{1}{2}, \phi + \frac{1}{2}])$.

If $E \notin Q(\phi)$, then τ-Harder-Narasimhan filtration $\underset{\underset{A_1}{\shortparallel}}{E_1} \longrightarrow E_2 \longrightarrow \dots \longrightarrow E_n = E$

Since $P((\phi - \frac{1}{2}, \phi + \frac{1}{2}])$ is a heart, $E_1 \hookrightarrow E$ is a monomorphism (in this heart). Furthermore $Z = W \Rightarrow Z(A)$ has bigger place than $Z(E)$ (by definition of τ-Harder-Narasimhan filtration) $\Rightarrow\!\Leftarrow$ since E is semistable in this heart.