Bridgeland Stability Conditions   Oct 8/2013.

$\mathcal{D}$ triangulated.

Def: A BSC on $\mathcal{D}$ is (Z, P) or $(Z, \heartsuit)$.
slicing

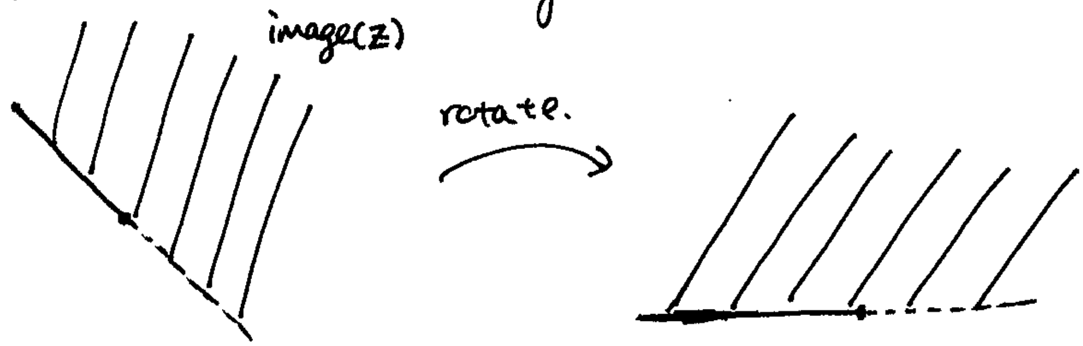
Last time:

Prop: Given (Z,P). then $\forall\, \phi_0 \in \mathbb{R}$, $P((\phi_0, \phi_0+1])$ is a heart of a bounded t-structure on $\mathcal{D}$.

$$\begin{tikzcd} \{(Z,P)\} \arrow[r, bend left, "\heartsuit = P((0,1])"] & \{(Z,\heartsuit)\} \arrow[l, bend left, "\substack{P(\phi+n) := P(\phi)[n] \\ \phi \in (0,1]}"] \end{tikzcd}$$

Prop: $\forall\, \phi_0 \in \mathbb{R}$, Z defines a BSC on $P((\phi_0, \phi_0+1])$.

Proof: Really, we should rotate Z by $e^{-i\pi\phi_0}$.



(0) is satisfied.

Now, Recall: $Z: \mathrm{ob}\,\mathcal{C} \to \mathbb{C}$ is a BSC on $\mathcal{C}$ if

(0) $\mathrm{image}(Z) \subset \overline{\mathbb{H}} \setminus \mathbb{R}_{>0}$

(1) $Z(E) = 0 \Rightarrow E \cong 0$.

(2) $0 \to A \to B \to C \to 0 \Rightarrow Z(B) = Z(A) + Z(C)$.

(3) H-N property.

(1) Let $0 \neq E \in P((\phi_0, \phi_0+1])$.
E has H-N filtration by $A_i \in P(\varphi_i)$ by (s|2).
$Z(A_i)$ are all in the upper half plane.
so $\sum Z(A_i) \neq 0$.

(2) obvious b/c a SES in a ♡ is an exact $\triangle$ in D all of whose objects are in ♡.

(3) obvious by the definition of slicing.

$\blacksquare$.

<u>Proposition</u>: If $A \in \mathrm{ob}\, P(\phi)$. $\phi \in (\phi_0, \phi_0+1]$.
then A is a Z-semistable object in $P((\phi_0, \phi_0+1])$.
The converse is also true.

<u>Proof</u>: WLOG $\phi_0 = 0$.
Let $\phi(E) > \phi(A)$. $E \in P((0,1])$.
and consider any map $f: E \to A$.
By H-N filtration on E,

$$E_1 \to E_2 \to \cdots \to E_n \xrightarrow{=} E$$

$\uparrow$ semi-stable.
and $\phi(E_1) > \phi(E)$.

so the composition $E_1 \hookrightarrow E \xrightarrow{f} A$ is 0.
(since E_1, A are semi-stable, and $\phi(E_1) > \phi(A)$.
so f has kernel.).

<u>on the other hand</u>, let
$A \in P((0,1])$ be Z-semistable. (In the sense of abelian categories).

If A is not its own H-N filtration, $\exists\ E_1 \hookrightarrow A$ with $\phi(E) > \phi(A)$. Contradiction. $\blacksquare$.

Topology on Stab(D)

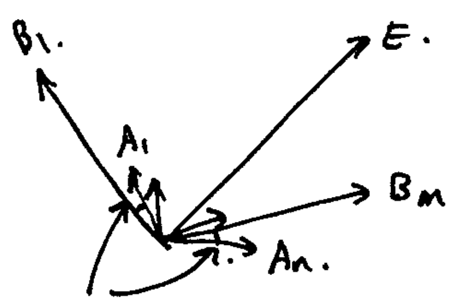
Def: Let P be a slicing for D.
$\forall E$,

$$\begin{tikzcd} 0 \arrow[r] & E_1 \arrow[r] \arrow[d, equal] & \cdots \arrow[r] & E_n = E \arrow[d] \\ & A_1 & \cdots\cdots & A_n. \end{tikzcd}$$

Let $\phi_P^+(E) := \phi_P(A_1)$.
$\phi_P^-(E) := \phi_P(A_n)$.

Example: $E \in P([\phi_P^-(E), \phi_P^+(E)])$.

Def: Given two slicings P, Q, define
$$\mathrm{dist}(P, Q) := \sup_{0 \neq E \in \mathrm{ob}D} \{ |\phi_P^+(E) - \phi_Q^+(E)|, |\phi_P^-(E) - \phi_Q^-(E)| \}.$$



$< \mathrm{dist}(P, Q)$.

<u>Prop:</u> distance satisfies :

- $\mathrm{dist}(P, Q) = 0 \Rightarrow P = Q$. ($P(\phi) = Q(\phi)$ for all ϕ).
- $\mathrm{dist}(P, Q) = \mathrm{dist}(Q, P)$.
- $\mathrm{dist}(P, Q) + \mathrm{dist}(Q, R) \geq \mathrm{dist}(P, R)$.

<u>Remark:</u> distance is a generalized metric. (because ∞ is allowed).

<u>Proof:</u>
- $\phi_Q^+(E) = \phi_P^+(E)$ $\quad$ (if $E \in P(\phi)$.)

$$\phi_Q^-(E) = \phi_P^-(E)$$

$\Rightarrow E \in Q(\phi)$.

- obvious.
- obvious.

<u>Remark</u>: Two slicings could have $P((0,1]) \cong Q((0,1])$ but still be distinct.

<u>Remark</u>: If $\mathrm{dist}(P, Q) < \varepsilon$, then
$P(\phi) \subset Q(\phi - \varepsilon, \phi + \varepsilon)$, and
$Q(\phi) \subset P(\phi - \varepsilon, \phi + \varepsilon)$.

<u>Fact:</u> (Bridgeland).

$$\mathrm{dist}(P, Q) = \inf \left\{ \varepsilon \geq 0 \;\middle|\; \begin{array}{c} Q(\phi) \subset P([\phi - \varepsilon, \phi + \varepsilon]) \\ \forall \phi \in \mathbb{R} \end{array} \right\}.$$

— What about Z? What's the topology on the set of functions $Z : K_0(D) \longrightarrow \mathbb{C}$?

Two approaches:

(1) (Bridgeland).
Demand (Z, P) be "locally finite"
i.e. $\forall \phi \in \mathbb{R}$, $\exists \varepsilon > 0$ such that
$P(\phi - \varepsilon, \phi + \varepsilon)$ is a finite length category.

⚠ $P((\phi-\varepsilon, \phi+\varepsilon))$ may not be abelian, they are quasi-abelian.
so you can still discuss sequences of subobjects and quotients.

(2) (Kontsevich - Soibelman).
Fix a lattice $\Gamma \cong \mathbb{Z}^N$, N finite.
(Γ could have torsion, but we ignore this).
Also fix a map

$$\begin{array}{c} K_0(\mathcal{D}) \\ {\scriptstyle cl}\downarrow \\ \Gamma \end{array}.$$

If we restrict to (Z, P), where Z factors

$$\begin{tikzcd} K_0(\mathcal{D}) \arrow[r, "Z"] \arrow[d, "cl"'] & \mathbb{C} \\ \Gamma \arrow[ur, dashed, "Z"'] & \end{tikzcd}.$$

Then we can induce a topology from
$\Gamma^\vee = \mathrm{hom}(\Gamma, \mathbb{C})$.
↑ finite dimensional.

Fix metric on $\Gamma \otimes \mathbb{R} \cong \mathbb{R}^N$ to define the norm

$$\|Z - W\| = \sup_{\gamma \in \Gamma} \left\{ \frac{|Z(\gamma) - W(\gamma)|}{\|\gamma\|} \right\}.$$

Now we have a metric!

Given $\sigma = (Z, P)$, $\tau = (W, Q)$.

$$d(\sigma, \tau) = \max \{ \operatorname{dist}(P, Q), \| Z - W \| \}.$$

<u>Example</u>: Let X be a smooth proj. curve $/ \mathbb{C}$.

$\mathcal{C} = \operatorname{coh}(X)$.

$\exists$ Mukai pairing

$$\operatorname{ob}\mathcal{C} \times \operatorname{ob}\mathcal{C} \to \mathbb{Z}.$$

$$(E, F) \longmapsto \Sigma (-1)^i \underset{\underset{\dim \operatorname{Ext}^i(E,F).}{\uparrow}}{\operatorname{ext}^i}(E, F)$$

Define the <u>numerical Grothendieck group</u> to be

$K_0(\mathcal{C}) / \{ E \mid \langle E, - \rangle \equiv 0 \}$.

We take Γ to be this numerical Grothendieck group.

If X is an elliptic curve,

$\Gamma \cong \mathbb{Z}^2$.

$$\begin{tikzcd} \Gamma & \cong & \mathbb{Z}^2 & (\deg E, \operatorname{rk} E). \\ K_0(\mathcal{C}) \arrow[u] \arrow[urr] & [E]. \arrow[urr, leftrightarrow] \end{tikzcd}$$

<u>Example</u>: Let M be a C-Y 3-fold. $\mathcal{C} = \operatorname{Fuk}(M)$.

$\operatorname{ob}\mathcal{C}$ are Lagrangians. (exact or monotone).

$L \subset M$. $\dim_{\mathbb{R}} L = \frac{1}{2} \dim_{\mathbb{R}} M$.

$\omega|_L \equiv 0$. send

$$\begin{tikzcd} {[L]} \arrow[d] & K_0(\mathcal{C}) \arrow[d] \\ {[L].} & H_3(M; \mathbb{Z}) \cong \Gamma. \end{tikzcd}$$

maybe $H^3(M; \mathbb{Z})$.

This should be the charge lattice for some BSC on $\operatorname{Fuk}(M)$.

Fact: In this example, one can show

$$\inf \left\{ \frac{\left|\int_L \Omega^{3,0}\right|}{\underbrace{\|L\|}_{\mathrm{vol}\, L}} ,\ L \text{ special} \right\} > 0.$$

Def: $\sigma = (Z, P)$ is said to satisfy the <u>support property</u> if $\inf \left\{ \frac{|Z(E)|}{\|E\|} ,\ E \text{ semistable} \right\} > 0$.

Remark: Not all BSC satisfy

(1) Local finiteness.

(2) support property.

Ex: $\mathcal{C} = \mathrm{coh}(X)$, X smooth proj curve $/\mathbb{C}$.

$$\begin{array}{ccc} [E] & & K_0(\mathcal{C}) \longrightarrow \mathbb{C} \\ \downarrow & & \downarrow \quad \nearrow \\ (\deg, \mathrm{rk}). & & \mathbb{Z}^2. \end{array}$$

fixed element of $\overline{\mathbb{H}} \setminus \mathbb{R}_{>0}$.

$$[E] \longmapsto Z(\deg E + \alpha\, \mathrm{rk}\, E).$$

α irrational.

next time.

Exercise: show $\mathrm{Coh}(\mathbb{P}^1)$ is not finite length.

<u>Def</u>: Fix $cl: K_0(D) \longrightarrow \Gamma$. Let $Stab(D)$ be the space of $\sigma = (Z, P)$ such that

- Z factors through Γ.
- Z satisfies the support property.

<u>Thm</u>: The map

$$Stab(D) \longrightarrow hom(\Gamma, \mathbb{C}).$$
$$(Z, P) \longmapsto Z.$$

is a local homeomorphism.

<u>Cor</u>: $Stab(D)$ is a complex manifold.

<u>Lemma</u>: $(Z, P) \longrightarrow Z$ is a local injection.

<u>proof</u>: Let $\sigma = (Z, P)$, $\tau = (W, Q)$.

we will show if $d(\sigma, \tau) < \frac{1}{4}$ and $Z = W$, then $P = Q$.

(i.e. $P(\phi) = Q(\phi), \forall \phi$).

$P(\phi) \subset Q(\phi)$:

Let $E \in P(\phi)$. Since it is semistable,

$\phi_P^+(E) = \phi(E) = \phi_P^-(E)$.

By the definition of distance,

$$\begin{aligned} P(\phi) &\subset Q((\phi - \tfrac{1}{4}, \phi + \tfrac{1}{4})). \\ &\subset P((\phi - \tfrac{1}{2}, \phi + \tfrac{1}{2})). \\ &\subset P((\phi - \tfrac{1}{2}, \phi + \tfrac{1}{2}]). \end{aligned}$$

If $E \notin Q(\phi)$, the τ H-N filtration

$$E_1 \longrightarrow E_2 \longrightarrow \cdots \longrightarrow E_r = E.$$

$\overset{\shortparallel}{A_1}$. (semi-stable)

Since $P((\phi-\frac{1}{2}, \phi+\frac{1}{2}])$ is a heart, $E_1 \hookrightarrow E$ is a monomorphism (in this heart).

Moreover, $Z = W$. $\Rightarrow$ $Z(A)$ has a bigger phase than $Z(E)$.
(by the defn of Σ H-N filtration).

Contradiction, because E is ss in this heart.

$\blacksquare$.