Equivalence of definitions, CONT'D

Last time we proved

Prop'n $\forall \phi_0 \in \mathbb{R}$,

$$P((\phi_0, \phi_0+1]) \subset D$$

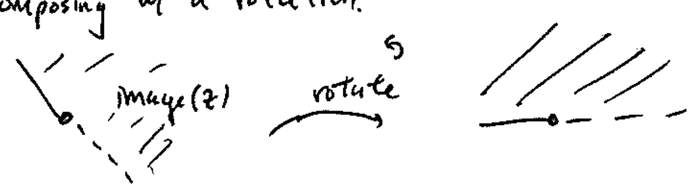
is a heart of a bounded t-structure on D.

Recall $P(I)$ is the full subcategory of those E whose H-N factors have phase in I.

Now let's prove

Prop'n $\forall \phi_0 \in \mathbb{R}$, Z defines a Bridgeland Stability Condition on $P((\phi_0, \phi_0+1])$.

Pf Of course, one should "rotate" Z by post-composing w/ a rotation.



So set Z to be $e^{-i\phi_0\pi} Z$.

Then (0) is satisfied.

(1) : $Z(\underset{E \neq 0}{\text{Every object}})$ is a sum of $Z(A_i)$, its H-N factors.

Each $Z(A_i) \neq 0$ since $Z(A_i) = m e^{i\pi\phi_i}$

By (0), no sum of such vectors can be zero.

(2) By characterization of SES in a heart as triangles that live in a heart

(3) As in (2), and by definition of slicing. //

Now all we need to do is prove (and used implicitly above) that any

$$A \in_{ob} \mathcal{P}(\phi), \quad \phi \in (0,1]$$

is a Z-semistable object in $\mathcal{P}((0,1])$ (as in the definition of semistability for ~~the~~ abelian categories).

Pf. Let $\overset{\in (0,1]}{\phi(E)} > \phi(A)$ and consider any morphism $f: E \to A$.

By H-N filtration, we have

$$E_1 \hookrightarrow E \xrightarrow{f} A.$$

Since $\phi(E_1) > \phi(E) > \phi(A)$, (SL1) says this composition is zero. But $E_1 \hookrightarrow E$ is mono, so f must have a kernel. //

On the other hand, let $A \in P((0,1])$ be semistable for Z, in the sense of abelian categories.

If A is NOT its own H-N filtration, $\exists$ $E_1 \hookrightarrow A$ with $\phi(E_1) > \phi(A)$ by (SI 2). But that's impossible by (SI 2).

So $A \in P(\phi(A))$.

This shows

$$\begin{tikzcd}
\{(P, Z)\} \arrow[rr, bend left, "\heartsuit = P((0,1])"] & & \{(\heartsuit, Z)\} \arrow[ll, bend left, "P(\phi+N) := P(\phi)[N]"]
\end{tikzcd}$$

are inverse operations.

[Topology on Stab(D)]

First, we define a topology on the set of slicings.

<u>Defn</u>. Let P be a slicing for D.
For every object E w/ H-N filtration

$$\begin{tikzcd}
0 \arrow[r] & E_1 \arrow[r] \arrow[d, equal] & E_2 \arrow[r] & \cdots \arrow[r] & E_n = E \arrow[d] \\
& A_1 & & & A_n
\end{tikzcd}$$

let $\phi_P(E) = \phi(A_1)$, $\phi^{+}_{P}(E) = \phi(A_n)$.

ϕ^{-}_{P} $\qquad\qquad$ ϕ^{+}_{P}

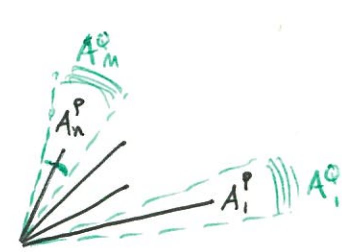
So for instance,

$$E \in \mathcal{P}([\phi_{\mathcal{P}}^{-}(E), \phi_{\mathcal{P}}^{+}(E)]).$$

Defn Given two slicings $\mathcal{P}$ and $\mathcal{Q}$, define

$$\operatorname{dist}(\mathcal{P},\mathcal{Q}) := \sup_{0 \neq E \in \operatorname{ob} \mathcal{D}} \left\{ |\phi_{\mathcal{P}}^{+}(E) - \phi_{\mathcal{Q}}^{+}(E)|, \ |\phi_{\mathcal{P}}^{-}(E) - \phi_{\mathcal{Q}}^{-}(E)| \right\}.$$

$$\in [0, \infty].$$



Propn dist satisfies

- $\operatorname{dist}(\mathcal{P},\mathcal{Q}) = 0 \implies \mathcal{P} = \mathcal{Q}$
- $\operatorname{dist}(\mathcal{P},\mathcal{Q}) = \operatorname{dist}(\mathcal{Q},\mathcal{P})$
- $\operatorname{dist}(\mathcal{P},\mathcal{Q}) + \operatorname{dist}(\mathcal{Q},\mathcal{R}) \geq \operatorname{dist}(\mathcal{P},\mathcal{R})$

where ∞ is the maximal element of $[0,\infty]$ and $\infty +$ anything $= \infty$.

Rmk This is called a generalized metric.

Pf

- take E to be semistable in $\mathcal{P}$, then

  since dist=0

  $$\phi_{\mathcal{Q}}^{+}(E) = \phi_{\mathcal{P}}^{+}(E) = \phi_{\mathcal{P}}^{-}(E) = \phi_{\mathcal{Q}}^{-}(E)$$

  $\implies E \in \mathcal{Q}(\phi)$. since $E \in \mathcal{P}(\phi)$.

  vice versa.

- obvious by symmetry of $|a-b|$.
- obvious, since $\forall E$,

$$|\phi_{\mathcal{P}}^{+}(E) - \phi_{\mathcal{Q}}^{+}(E)| + |\phi_{\mathcal{Q}}^{+}(E) - \phi_{\mathcal{R}}^{+}(E)| \geq |\phi_{\mathcal{P}}^{+}(E) - \phi_{\mathcal{R}}^{+}(E)|$$

by triangle inequality on $\mathbb{R}$.

Rmk If $\operatorname{dist}(P,Q)=\infty$, P and Q are in different connected components. (By definition of metric topology.)

Rmk While $P((0,1])$ and $Q((0,1])$ may stay constant, slicings themselves can be distinct — for instance, for a different interval $P((\phi_0,\phi_0+1])$ may look different than $Q((\phi_0,\phi_0+1])$.

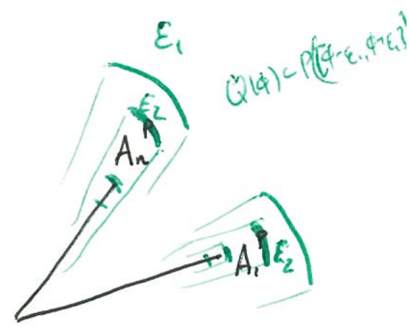
Rmk If $\operatorname{dist}(P,Q)<\varepsilon$ $(=)$ clearly

$$P(\phi) \in Q((\phi-\varepsilon,\phi+\varepsilon)) \quad ([\ \ ]) \quad \forall \phi$$

and

$$Q(\phi) \subset P((\phi-\varepsilon,\phi+\varepsilon)) \quad ([\ \ ]) \quad \forall \phi.$$

In fact,



Lemma If P and Q are slicings,

$$\operatorname{dist}(P,Q) = \inf\{\varepsilon \geq 0 \mid Q(\phi)\subset P([\phi-\varepsilon,\phi+\varepsilon]) \ \forall \phi\in\mathbb{R}\}.$$

Pf Let RHS be $d'(P,Q)$. By remark,

$$d(P,Q)\leq\varepsilon \to d'(P,Q)\leq\varepsilon.$$

The rest: Some other time.

Now we show

$$d'(\mathcal{P},\mathcal{Q}) \le \varepsilon \implies \begin{array}{c} |\phi^+_Q(E) - \phi^+_P(E)| \le \varepsilon \\ \text{and} \\ |\phi^-_Q(E) - \phi^-_P(E)| \le \varepsilon \end{array} \qquad \forall \underset{\neq 0}{E} \in \mathrm{ob}\,\mathcal{D}.$$

<u>Lemma</u>. If $E \in Q((-\infty, \psi])$

then $E \in P((-\infty, \psi + \varepsilon])$.

<u>Pf</u> Otherwise, $\exists$ P-H-N filt

$$A_1 \to \cdots \longrightarrow A_n = E$$

w/ $\phi_P(A_1) > \psi + \varepsilon$.

So how do we define a topology on set of $Z : K_0(D) \to \mathbb{C}$?

Two approaches:

(1) (Bridgeland) Demand (Z, P) be "locally finite."

This means $\forall \phi \in \mathbb{R}$, $\exists$ $\varepsilon > 0$ s.t.

$$P(\phi - \varepsilon, \phi + \varepsilon)$$

is a "finite length" category.

⚠ Note $P(\phi-\varepsilon, \phi+\varepsilon)$ may not be abelian;

It's "quasi-abelian" so you still have good notion of subobjects and quotients.

We won't get into this.

(2) (Kontsevich-Soibelman)

Fix a lattice $\Gamma \cong \mathbb{Z}^N$, N finite,

and a homomorphism

$$\begin{array}{c} K_0(D) \\ \mathrm{cl} \downarrow \\ \Gamma \end{array}$$

Γ is called the charge lattice.

If we restrict attention to those Z that factor through Γ, we have a map

$$\begin{aligned} \{(Z, P)\} &\longrightarrow \hom(\Gamma, \mathbb{C}) \\ (Z, P) &\longmapsto Z \end{aligned}$$

identifying $\mathrm{ob}\,\mathcal{D} \xrightarrow{Z} \mathbb{C}$ with $\Gamma \xrightarrow{Z} \mathbb{C}$.

So we can induce a topology from $\Gamma^\vee = \hom(\Gamma, \mathbb{C})$.

Fix a metric on $\Gamma \otimes \mathbb{R} \cong \mathbb{R}^N$, any will do.

This defines a norm on $\Gamma^\vee$ by

$$\|Z - W\| := \sup_{\gamma \in \Gamma} \left\{ \frac{\| Z(\gamma) - W(\gamma) \|}{\|\gamma\|} \right\}.$$

Now we have a metric:

if $\begin{aligned} \sigma &= (Z, P) \\ \tau &= (W, Q) \end{aligned}$ then $d(\sigma, \tau) = \sup \{ \mathrm{dist}(P, Q), \|Z - W\| \}$.

Example. Let X be a complex projective curve.

$\exists$ a Mukai pairing

$$\mathrm{ob}X \times \mathrm{ob}X \longrightarrow \mathbb{Z}$$
$$(E\ ,\ F) \longmapsto \langle E,F\rangle := \sum (-1)^i \mathrm{ext}^i(E,F).$$

Let the numerical Grothendieck group be

$$K_0(\mathrm{Coh}(X)) \Big/ \{E \mid \langle E,-\rangle \equiv 0\}.$$

We can take Γ to be the above group.

Exer. Show Mukai pairing descends to K_0.

Pf. LES of Ext.

$$\begin{array}{c} E_0 \longrightarrow E_1 \\ \nwarrow \quad \swarrow \\ E_2 \end{array} \Longrightarrow \cdots \to \mathrm{Ext}^{i-1}(E_2,F) \xrightarrow{d^{i-1}} \mathrm{Ext}^i(E_0,F) \xrightarrow{a^i} \mathrm{Ext}^i(E_1,F) \xrightarrow{b^i} \mathrm{Ext}^i(E_2,F) \xrightarrow{d^i} \cdots$$

$$\sum (-1)^i \dim \mathrm{Ext}^i(E_1,F) \longleftrightarrow \sum (-1)^i \dim \mathrm{Ext}^i(E_0,F) + (-1)^i \dim \mathrm{Ext}^i(E_2,F)$$

$$\|$$

$$\sum (-1)^i (\mathrm{image}(a^i) + \ker(\partial^i)) \quad = \pm \mathrm{im}\, a^{i-1} \pm \ker a^{i-1} \pm \mathrm{im}\, d^{i-1} \pm \ker d^{i-1}$$

cancel!

$$\mp \mathrm{im}\, a^i \mp \ker a^i \mp \mathrm{im}\, d^i \mp \ker d^i$$

Example Let M be a Calabi-Yau 3-fold,

$\mathcal{C} = \mathrm{Fuk}(M)$ the Fukaya category.

Objects are Lagrangians, $L \subset M$, $\dim_{\mathbb{R}} L = \frac{1}{2} \dim_{\mathbb{R}} M$

So send

$$L \longmapsto [L] \in H_3(M, \mathbb{Z}).$$

This should be the charge lattice.

$H_3(M, \mathbb{Z}) = \Gamma$

In this last example, one can prove:

$$\inf \left\{ \frac{\left| \int_L \Omega^{3,0} \right|}{\| [L] \|} \ , \ L \text{ semistable (i.e., special) Lagrangian} \right\} > 0.$$

This inspires

Defn $\sigma = (Z, P)$ satisfies the support property if

$$\inf \left\{ \frac{|Z(E)|}{\| E \|} \ , \ E \text{ semistable} \right\} > 0.$$

Defn Fix $cl : K_0(D) \to \Gamma$.

Let

$$\mathrm{Stab}(D)$$

be the space of $\sigma = (Z, P)$

such that

- Z factors through cl

- Z satisfies the support condition.

Thm.

$$\begin{aligned} \mathrm{Stab}(D) &\longrightarrow \hom(\Gamma, \mathbb{C}) \\ \sigma(Z, P) &\longmapsto Z \end{aligned}$$

is a local homeomorphism.

Cor $\mathrm{Stab}(D)$ is a complex manifold.

Take charts via the local homeomorphisms.

Then transition functions $\underset{\hom(\Gamma, \mathbb{C})}{\underset{\wedge}{U}} \xrightarrow{\sim} \underset{\hom(\Gamma, \mathbb{C})}{\underset{\wedge}{V}}$

are identity.

Lemma $\mathrm{Stab}(D) \longrightarrow \hom(\Gamma, \mathbb{C})$

is a local injection.

Pf. Let $\sigma = (Z, P)$

$\tau = (W, Q)$.

We show that if

$$d(\sigma, \tau) < \tfrac{1}{4} \quad \text{and} \quad Z = W$$

then $P = Q$; ie, $P(\phi) = Q(\phi)\ \forall \phi$.

$\boxed{P(\phi) \subset Q(\phi)}$ (other direction, $Q(\phi) \subset P(\phi)$, by symmetry.)

Let $E \in P(\phi)$. Since E is semistable,

$$\phi_P^+(E) = \phi_P(E) = \phi_P^-(E).$$

By definition of dist, note

$$P(\phi) \subset Q((\phi - \tfrac{1}{4}, \phi + \tfrac{1}{4})) \subset P((\phi - \tfrac{1}{2}, \phi + \tfrac{1}{2}))$$

$$\subset P((\phi - \tfrac{1}{2}, \phi + \tfrac{1}{2}]).$$

If $E \notin Q(\phi)$, consider its τ-H-N filtration

$$\underset{\text{semistable}}{A} \to E_2 \to \cdots \to E_n = E.$$

Since all triangles are in $Q((\phi - 1/4, \phi + 1/4)) \subset P((\phi - 1/2, \phi + 1/2])$,
$A \hookrightarrow E$ is a monomorphism in $P((\phi - 1/2, \phi + 1/2])$.

Further, $Z = W \Rightarrow Z(A)$ has bigger phase than $Z(E)$.
(By defn of a W-H-N filtration.)

This contradicts E being semistable in $P((\phi - 1/2, \phi + 1/2])$. //