Bridgeland Stability Conditions. Oct 10, 2013.

Thm: $\mathrm{Stab}(D) \longrightarrow \mathrm{hom}(\Gamma, \mathbb{C})$ is a local homeomorphism.
$(Z, P) \longmapsto Z$.

The objects in play: • fix a finitely generated abelian group Γ.
• fixed
$$\begin{tikzcd} K_0(D) \arrow[d, two heads, "cl"'] \\ \Gamma \end{tikzcd}$$
• fixed norm $\|\cdot\|$ on $\Gamma \otimes \mathbb{R}$.

Def: $\mathrm{Stab}(D) = \left\{ (Z, P) \;\middle|\; \begin{array}{l} \cdot\, Z \text{ factors through } \begin{tikzcd} K_0(D) \arrow[r, "Z"] \arrow[d, "cl"'] & \mathbb{C} \\ \Gamma \arrow[ur, dashed, "Z"'] & \end{tikzcd} \\ \cdot\, Z \text{ satisfies the support property.} \end{array} \right\}$

Def: Z satisfies the <u>support property</u> if $\inf \left\{ \dfrac{|Z(E)|}{\|E\|},\ E \text{ s.s} \right\} > 0$.

Topology on $\mathrm{Stab}(D)$ is given by:

- $\mathrm{dist}(P, Q) := \sup\limits_{E \in \mathrm{ob}D} \left\{ |\phi_P^+(E) - \phi_Q^+(E)|,\ |\phi_P^-(E) - \phi_Q^-(E)| \right\}$.
- $\|Z - W\| := \sup\limits_{cl(E) \neq 0} \left\{ \dfrac{|Z(E) - W(E)|}{\|E\|} \right\}$.

If $\sigma = (Z, P)$, $\tau = (W, Q)$,
$d(\sigma, \tau) := \max\{\mathrm{dist}(P, Q),\ \|E - W\|\}$.

<u>Last time:</u>

Prop: $\mathrm{stab}(D) \longrightarrow \mathrm{hom}(\Gamma, \mathbb{C})$ is locally injective.
The proof only required
$$\begin{aligned} P(\phi) &\subset Q((\phi - \tfrac{1}{4}, \phi + \tfrac{1}{4})) \\ &\subset P((\phi - \tfrac{1}{2}, \phi + \tfrac{1}{2})) \\ &\subset P((\phi - \tfrac{1}{2}, \phi + \tfrac{1}{2}]). \end{aligned}$$

<u>Thm</u> (Bridgeland).
Let $Stab^{loc}(D)$ be the space of all <u>locally finite</u> stability conditions.
Then $\forall$ connected ~~space~~ component $\Sigma \subset Stab^{loc}(D)$,
$\exists$ a linear subspace $V(\Sigma) \subset K_0(D)^\vee$ s.t the forgetful map
$\Sigma \longrightarrow hom(K_0(D), \mathbb{C})$ is a local homeomorphism onto $V(\Sigma)$.

<u>Lemma A</u>: The function $s: Stab(D) \longrightarrow \mathbb{R}_{>0}$ is continuous.

$$\sigma \longmapsto \inf_{E \ s.s} \left\{ \frac{Z(E)}{\|E\|} \right\}$$

<u>Lemma B</u>: Fix $\sigma=(Z,P)$. $\forall\ \Sigma$ small enough, assume $W: \Gamma \to \mathbb{C}$ such that

- $\|W-Z\| < \varepsilon \cdot S(\sigma)$.
- $im(W) = im(Z)$ (or $Re(W) = Re(Z)$)

Then $\exists$ slicing Q such that $\tau=(W,Q) \in Stab(D)$ and $dist(P,Q) < \varepsilon$.

<u>Proof of Theorem</u>:

$hom(\Gamma, \mathbb{C}) \cong hom(\Gamma, \mathbb{R}) \oplus hom(\Gamma, \mathbb{R})$.

Take $V = \{ re\ w \mid \|z-w\| < \varepsilon \cdot S(\sigma) \}$. $\subset hom(\Gamma, \mathbb{R})$.

we have a function $P: V \longrightarrow \mathbb{R}$

$$re\, w \mapsto \varepsilon \cdot S(\tau).$$

and P is continuous. so

$$\bigcup_{re\, w \in V} \{re\, w\} \times B_P(im\, w) \subset hom(\Gamma, \mathbb{R}) \oplus hom(\Gamma, \mathbb{R})$$

is open. ∎

Proof of Lemma A: $\sigma = (Z, P)$, $\tau = (W, Q)$.

we will show $\forall \delta > 0$, $\exists$ ε' s.t $d(\tau, \sigma) < \varepsilon'$.

$\Rightarrow$ $(s(\sigma) + \delta >)\ s(\tau) > (s(\sigma) - \delta)$

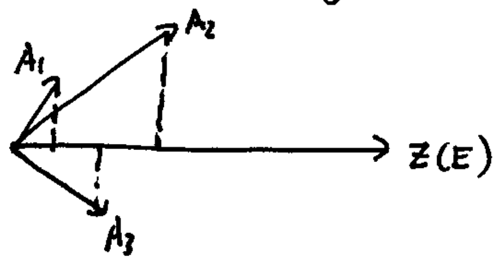
Assume $dist(P, Q) < \varepsilon$, and fix E semi-stable.

By definition, we know that letting A_i be the H-N factors for E (wrt σ), $|\phi_Z(A_i) - \phi_Z(E)| < \varepsilon$.

For $2\pi\varepsilon < \pi/2$,

$$|Z(E)| = \sum |Z(A_i)| \cos(2\pi\alpha_i)$$

where α_i is the angle between $Z(E)$ and $Z(A_i)$.



on the otherhand, $\alpha_i < \varepsilon$.

so $\cos(2\pi\alpha_i) > \cos(2\pi\varepsilon)$

$$\Rightarrow |Z(E)| > \sum |Z(A_i)| \cos(2\pi\varepsilon)$$

$$\Rightarrow \frac{|Z(E)|}{\|E\|} > \cos(2\pi\varepsilon) \cdot s(\sigma).$$

$$\left(\frac{\sum |Z(A_i)|}{\underset{\sum A_i}{\overset{\|E\|}{\shortparallel}}} \geqslant \frac{\sum Z(A_i)}{\sum \|A_i\|} \geqslant \frac{\sum s(\sigma) \|A_i\|}{\sum \|A_i\|} \right)$$

Now assume $\|W - Z\| < \varepsilon$. Then

$$\frac{|W(E)|}{\|E\|} \geqslant \frac{|Z(E)|}{\|E\|} - \frac{\|W(E) - Z(E)\|}{\|E\|}$$

$$\geqslant \frac{|Z(E)|}{\|E\|} - \varepsilon.$$

$$> \cos(2\pi\varepsilon) \cdot s(\sigma) - \varepsilon$$

This is true for $\forall$ E w-s.s. so

$$S(\mathcal{I}) = \inf \frac{|W(E)|}{\|E\|} \geq \cos(2\pi\varepsilon)\cdot S(\sigma) - \varepsilon.$$

And $\forall\, \delta > 0$, we can find $\varepsilon > 0$ such that

$$1 - a\varepsilon^2 \approx \cos(2\pi\varepsilon)\cdot \text{Blah} - \varepsilon > \text{Blah}\cdot\delta.$$

$\blacksquare$.

<u>useful inequality</u>:

(a) If E is σ-s.s, and $\|W - Z\| < \varepsilon \cdot S(\sigma)$. show that $\|W(E) - Z(E)\| < \varepsilon \cdot |Z(E)|$.

<u>Proof of Lemma B</u>:

Note: W defines a function $\operatorname{ob} P((0,1]) \longrightarrow \mathbb{C}$.

satisfying: (0)

(1)

(2) obvious.

(0): $\operatorname{image}(W) \subseteq \bar{H} \setminus \mathbb{R}_{>0}$.

If $\phi(E) \neq 1$, obviously $W(E)$ is in $\mathbb{H}$.

If $\phi(E) = 1$, and E is semi-stable in $P((0,1])$,

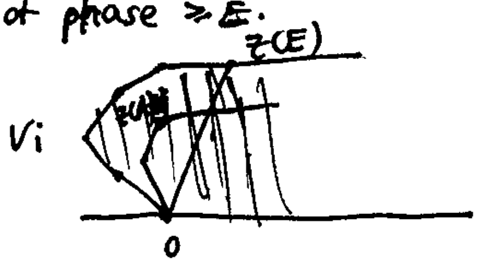
(a) $\Rightarrow$ $|\operatorname{Re} W(E) - \operatorname{Re} Z(E)| < \|Z(E)\|_\varepsilon = |\operatorname{Re} Z(E)|_\varepsilon$.

(1) is satisfied because if $E \neq 0$, then $W(E) \neq 0$.

<u>we need to show</u>: W satisfies the H-N property.

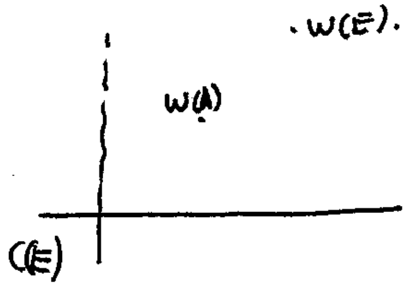
<u>step 1:</u> Let $\mathcal{H}_Z(E)$ be the convex hull of $\{Z(A) \mid A \subset E\}$.

Let $\mathcal{H}_Z^{\geq}(E) \subset \mathcal{H}_Z(E)$ be the part of $\mathcal{H}_Z(E)$ with vectors of phase $\geq E$.



$V_i = Z(A_i)$, A_i are H-N factors.

<u>Claim:</u> $\exists\ C(E) \in \mathbb{R}$ such that $Re(w(A)) \geq C(E)\ \forall A \subset E$.



<u>Proof:</u> examine $\mathcal{H}_Z(A)$ for $A \subset E$.

Fix $y = im\, Z(A)$. $w_i = Z$ (H-N factors B_i of A)

Then $\Sigma|w_i| \leq$ · $x(y)$, the real coord of point on $\partial\mathcal{H}_Z(E)$ of height y.

· $\ell(y)$ is length of PL path from 0 along $\partial\mathcal{H}_Z(E)$ to $(x(y), y)$.

So $Re(Z(A)) \geq \Sigma|w_i| + x(y) - \ell(y)$.

$\geq \Sigma|w_i| + C(E)$.

$x(y) - \ell(y)$ defines a continuous function ~~[illegible]~~ $[0, im\, Z(E)] \longrightarrow \mathbb{R}$.

Let $C(E)$ be minimum.

By inequality (a),

$$Re\ w(A) \geq Re\ Z(A) - \varepsilon \cdot \Sigma|w_i|.$$

$$\left(\begin{aligned} \|w(B_i) - Z(B_i)\| &= \|Re\, w(B_i) - Re\, Z(B_i)\| \\ &< \varepsilon \cdot \|Z(B_i)\| \end{aligned} \right).$$

$$Re\ w(A) \geq (\Sigma|w_i|)(1-\varepsilon) + C(E) \geq C(E).$$

<u>Step two:</u>

<u>Claim:</u> $\exists$ only finitely many $\gamma \in \Gamma$ such that

- $\exists A \subset E$, with $cl(A) = \gamma$, and
- $Z(A) \in \mathcal{H}_W^{\varepsilon}(E)$.

Using inequality from before,

$$\operatorname{Re} W(A) - C(E) \geq (1-\varepsilon)\sum |W_i| .$$

$$\Rightarrow \frac{\operatorname{Re} W(A) - C(E)}{1-\varepsilon} \geq \sum |W_i| .$$

By assumption that $W(A)$ lies to the left of $W(E)$,

$$\operatorname{Re} W(A) \leq \operatorname{Re} W(E).$$

$$D(E) := \frac{\operatorname{Re} W(E) - C(E)}{1-\varepsilon} \geq \sum |W_i|$$

$$\Rightarrow \|A\| \leq \frac{D(E)}{S(\sigma)}$$

$$\Rightarrow \frac{D(E)}{S(\sigma)} \geq \sum |W_i| \frac{\|A\|}{\underbrace{|\sum Z(B_i)|}_{= |Z(A)|}} .$$

$$\geq \sum |W_i| \frac{\|A\|}{\sum |Z(B_i)|}$$

$$= \|A\| .$$