We continue the proof of

Thm $\mathrm{Stab}(\mathcal{D}) \to \hom(\Gamma, \mathbb{C})$

is a local homeomorphism.

Recall the objects in play:

- A fixed homomorphism, surjective,

$$\begin{array}{c} K_0(\mathcal{D}) \\ {\scriptstyle cl}\downarrow \\ \Gamma \end{array}$$

where Γ is a finitely generated abelian group

- A norm $\|\cdot\|$ on $\Gamma \otimes \mathbb{R}$

- $\mathrm{Stab}(\mathcal{D}) = \{ (Z, P)$ s.t. $\cdot$ Z factors though Γ via cl

$$\begin{tikzcd} K_0(\mathcal{D}) \arrow[r, "Z"] \arrow[d, "cl"'] & \mathbb{C} \\ \Gamma \arrow[ur, dashed, "\exists"'] & \end{tikzcd}$$

$\cdot$ Z satisfies support property $\}$.

The support property says

$\|\cdot\|$ on $\mathbb{C}$

$$\inf_{E \text{ semistable}} \left\{ \frac{\|Z(E)\|}{\|E\|} \right\} > 0 .$$

$\|\cdot\|$ on $\Gamma\otimes\mathbb{R}$

We really mean $\|cl(E)\|$ by this notation.

The topology on $Stab(D)$ is given by the sup of two metrics:

- $$\operatorname{dist}(P,Q) := \sup_{E \in \operatorname{ob} D} \left\{ |\phi_P^+(E) - \phi_Q^+(E)|,\ |\phi_P^-(E) - \phi_Q^-(E)| \right\}$$

- $$\|Z - W\| = \cancel{\sup_{\gamma\neq 0}\left\{ \frac{\|Z(\gamma) - W(\gamma)\|}{\|\gamma\|} \right\}}$$

$$= \sup_{cl(E)\neq 0} \left\{ \frac{\|Z(E) - W(E)\|}{\|E\|} \right\} .$$

Given two stability conditions

$$\sigma = (Z, P) \qquad \tau = (W, Q)$$

we define:

$$d(\sigma,\tau) := \max\left(\operatorname{dist}(P,Q),\ \|Z-W\| \right).$$

Last time we proved

Prop'n $\mathrm{Stab}(\mathcal{D}) \to \hom(\Gamma, \mathbb{C})$ is a local injection.

This did NOT rely on any of the set-up. In general, we can actually prove

Thm (Bridgeland) If $\mathrm{Stab}^{\mathrm{loc}}(\mathcal{D})$ is the space of all $\overbrace{\text{stability}}^{\text{locally finite}}$ conditions, $\exists$ a linear subspace $V(\Sigma) \subset \hom(K_0(\mathcal{D}), \mathbb{C})$ (possibly ∞-dim.) $\forall$ connected components $\Sigma \subset \mathrm{Stab}^{\mathrm{loc}}(\mathcal{D})$, s.t. the forgetful map

$$\begin{aligned} \Sigma &\longrightarrow V(\Sigma) \\ (Z, P) &\longmapsto Z \end{aligned}$$

is a local homeomorphism.

Recall I told you that "support property" implies "locally finite."

The rest of our proof (not Bridgeland's theorem in full generality, but for ~~the~~ our support property) relies on two lemmas:

Lemma A: The function

$$\delta : Stab(D) \longrightarrow \mathbb{R}_{>0}$$
$$\sigma \longmapsto \inf_{E \text{ s.s.}} \left\{ \frac{\|Z(E)\|}{\|E\|} \right\}$$

is continuous.

Lemma B: Let $\varepsilon > 0$ such that

$$\cos(2\pi\varepsilon) - \varepsilon > 0,$$

and assume $W : \Gamma \to \mathbb{C}$ is a map s.t.

- $\|W - Z\| < \varepsilon \cdot \delta(\sigma)$, $\sigma = (Z, P)$.
- $\mathrm{im}(W) = \mathrm{im}(Z)$, or $\mathrm{Re}(W) = \mathrm{Re}(Z)$

Then $\exists$ slicing Q s.t.

$$(W, Q) = \tau \in Stab(D)$$

with dist$(P, Q) < \varepsilon$.

we'll verify this part online.

Pf of theorem:

$$\hom(\Gamma, \mathbb{C}) \longrightarrow \hom(\Gamma, \mathbb{R}) \oplus \hom(\Gamma, i\mathbb{R})$$

is a homeomorphism.

Given $\sigma = (Z, P)$, define a function

$$\rho : V \subset \hom(\Gamma, \mathbb{R}) \longrightarrow \mathbb{R} \qquad \left(V = \{W \mid \|Z - W\| < \varepsilon \cdot S(\sigma)\}\right).$$
$$\mathrm{re}\,W \longmapsto \varepsilon \cdot S(\tau)$$

This is continuous, so the set

$$\bigcup_{\mathrm{re}\,W \in V} \{\mathrm{re}\,W\} \times B_\rho(\mathrm{im}\,W) = \mathcal{U} \subset \hom(\Gamma, \mathbb{C})$$

is open.

To see this, choose any compact K around $\mathrm{re}\,Z \in V$.
Then $\rho|_K$ attains a minimum; fix $\varepsilon <$ this minimum.
Thus $\mathcal{U}$ contains the set (ball around $\mathrm{re}\,Z$) $\times$ (ball of radius ε around $\mathrm{im}\,Z$)
which is open. So $\forall (Z, P)$, we've exhibited open $\mathcal{U} \ni Z$ onto which we surject.

Pf of Lemma A. I'll just show $\forall \delta > 0$,

$$S(\tau) > S(\sigma) - \delta$$

for τ close enough to σ. By symmetry, this gives

$$S(\sigma) > S(\tau) - \delta$$

i.e.,

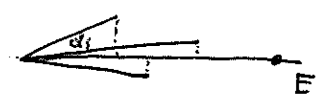
$$|S(\tau) - S(\sigma)| < \delta.$$

Assume $\mathrm{dist}(P, Q) < \varepsilon$, and E is τ-semi-stable. Then

$$Z(E) = \sum Z(A_i) \qquad A_i \text{ H-N factors for } E \text{ in } \sigma$$

~~For ε small enough, we have~~

Further, by definition, we know

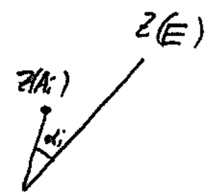


$$|\phi_Z(A_i) - \phi_Z(E)| < \varepsilon$$

since

$$Q(\phi_Q(E)) \subset P((\phi_Q(E) - \varepsilon, \phi_Q(E) + \varepsilon)).$$

For $|\pi\varepsilon| < \frac{\pi}{2}$, we have



$$|Z(E)| = \sum |Z(A_i)| \cos(2\pi\alpha_i)$$

and $\alpha_i < \varepsilon \Longrightarrow \cos(2\pi\alpha_i) > \cos(2\pi\varepsilon)$

$$\Longrightarrow |Z(E)| > \sum |Z(A_i)| \cos(2\pi\varepsilon).$$

$$\Longrightarrow \quad \frac{|Z(E)|}{\|E\|} > \frac{\sum |Z(A_i)| \cos(2\pi\varepsilon)}{\|E\|}$$

$$\geq \cos(2\pi\varepsilon) \frac{\sum |Z(A_i)|}{\sum_i \|A_i\|} \qquad \text{since } \|E\| \leq \sum \|A_i\|$$

$$\geq \cos(2\pi\varepsilon) \frac{\sum_i S(\sigma)\|A_i\|}{\sum \|A_i\|} \qquad \text{since } S(\sigma) \leq \frac{|Z(A_i)|}{|A_i|}$$

$$= \cos(2\pi\varepsilon) \cdot S(\sigma).$$

Now we assume $\|W - Z\| < \varepsilon$. Then

$$\frac{|W(E)|}{\|E\|} \geq \frac{|Z(E)|}{\|E\|} - \frac{\|W(E) - Z(E)\|}{\|E\|}$$

$$\geq \frac{|Z(E)|}{\|E\|} - \varepsilon \qquad \text{by defn of } \|W - Z\|$$

$$> \cos(2\pi\varepsilon) \cdot S(\sigma) - \varepsilon.$$

This is true $\forall$ E W-ss, so

$$S(\tau) = \inf_{E \text{ s.s.}} \frac{|W(E)|}{\|E\|} \geq \cos(2\pi\varepsilon) \cdot S(\sigma) - \varepsilon.$$

And we can always find $\varepsilon > 0$ s.t.

this is locally like
$t - \varepsilon - 2\pi t \varepsilon^2$.

$$\cos(2\pi\varepsilon) t - \varepsilon > t - \delta, \quad \forall \ \delta > 0.$$

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Pf of Lemma B When $\mathrm{im}(W) = \mathrm{im}(Z)$.

Note: W automatically satisfies

(0) (a) $\Rightarrow$ $|\mathrm{Re}\,W(A) - \mathrm{Re}\,Z(A)| < \|Z(A)\|\varepsilon$
$= |Z(A)|\varepsilon$
for $A \in P(1)$.

(2) by defn. already countable.

(1) by (0), since Z satisfied (1), and no vector in $H \setminus \mathbb{R}_{\geq 0}$ can sum to zero.

[NTS: H-N property, support property]

Step One: Let $\mathcal{H}_Z(E)$ be convex hull of $\{Z(A) \mid A \subset E\}$.

Let $\mathcal{H}_Z^{\geq}(E) \subset \mathcal{H}_Z(E)$ be intersection of $\mathcal{H}_Z(E)$ w/ all complex numbers w/ phase bigger than E.

Then $\exists\ C(E) \in \mathbb{R}_{(\geq 0)}$ s.t.

$$\mathrm{Re}\,W(A) \geq C(E) \quad \forall\ A \subset E.$$

Useful inequalities:

(a) If E σ-ss and $\|W - Z\| < \varepsilon \cdot S(\sigma)$ then $\|W(E) - Z(E)\| < \varepsilon |Z(E)|$.

Exercise:

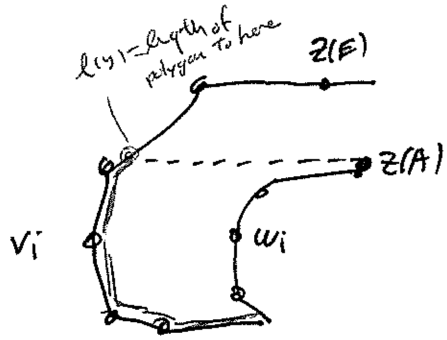
$$\frac{|W(E) - Z(E)|}{|E|} < S(\sigma)\,\varepsilon < \frac{|Z(E)|}{|E|}\cdot\varepsilon.$$

(b) If $\mathrm{im}\,W = \mathrm{im}\,Z$, then

$\mathrm{Re}\,(W(A)) \geq \mathrm{Re}\,(Z(A)) - \varepsilon|Z(A)|$

if A is Z-ss

Pf.



v_i = H-N factors of Z

w_i = " of A.

Fix $y = \operatorname{im} Z(A)$. Then

$$\sum |w_i| \leq l(y) + \operatorname{Re} Z(A) - x(y)$$

where

- $l(y)$ is length of piecewise linear path from origin, through v_i, to $x(y)$
- $x(y) = \partial \mathcal{H}_Z^{?}(E) \cap$ line of height y.

So $$\operatorname{Re} Z(A) \geq \sum |w_i| + x(y) - l(y).$$

But $x(y) - l(y)$ defines a continuous function from a compact interval

$$x(y) - l(y) : [0, \operatorname{im} Z(E)] \longrightarrow \mathbb{R}$$

so let $C(E)$ be its minimum.

$$\operatorname{Re} Z(A) \geq \sum |w_i| + C(E).$$

Let B_i be H-N factors of A (wrt Z)

so $\qquad w_i = Z(B_i)$.

Note

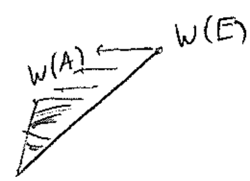
$$\mathrm{Re}\, W(B_i) \geq \mathrm{Re}\, Z(B_i) - \varepsilon |Z(B_i)| \quad \forall i$$

using useful inequality (b).

Summing over all i,

$$\begin{aligned}\mathrm{Re}\, W(A) &\geq \mathrm{Re}\, Z(A) - \varepsilon \textstyle\sum |w_i| \\ &\geq \textstyle\sum |w_i| + C(E) - \varepsilon \sum |w_i| \\ &\geq C(E) + (1-\varepsilon) \textstyle\sum |w_i| \qquad (*) \\ &\geq C(E) \quad \text{for } \varepsilon < 1.\end{aligned}$$

$$(*) \Rightarrow \quad \frac{\mathrm{Re}\, W(A) - C(E)}{1-\varepsilon} \geq \sum |w_i|.$$



If phase of $W(A) \geq$ phase $W(E)$, we have

$$D(E) := \frac{\mathrm{Re}\, W(E) - C(E)}{1-\varepsilon} \geq \sum |w_i|.$$

$$\frac{D(E)}{S(\sigma)} \geq \sum |w_i| \frac{\|A\|}{|\sum Z(B_i)|} \qquad \text{since } \sum Z(B_i) = Z(A)$$

$$\geq \sum |w_i| \frac{\|A\|}{\sum |w_i|} \qquad \Delta \text{ inequality}$$

$$\geq \|A\|.$$

Since $\|A\| := \|c\ell A\|$ is bounded by a # depending only on E, $\exists$ only finitely many $A \subset E$ with $\text{phase}\, W(A) \geq \text{phase}\, W(E)$.

Hence $\mathcal{H}_W^{\geq}(E)$ has only finitely many extremal pairs, hence W satisfies H-N property by our theorem from a few weeks ago.