בס"ד

280x (15/10)

Let's try to understand v.b. on $\mathbb{P}^1$:

$\mathbb{P}^1$ has open cover by $\mathbb{A}^1 = \operatorname{Spec} k[s]$

$\mathbb{A}^1 = \operatorname{Spec} k[t]$

and glueing $\mathbb{A}^1 - \{0\} = \operatorname{Spec} k[t, t^{-1}] \longrightarrow \operatorname{Spec} k[s, s^{-1}] = \mathbb{A}^1 - \{0\}$

$s \longmapsto t^{-1}$

A vector bundle is determined by glueing data. That is,

$$(u, v) \longmapsto (u^{-1}, A(u, u^{-1}) v)$$

nxn matrix with entries Laurent polynomial

Condition on A: $\det A(u, u^{-1}) \neq 0$ for $u \neq 0, \infty$.

Restrict to line bundles. Then $A(u, u^{-1}) =$ 1x1 matrix i.e. Laurent polynomial it does not vanish except possibly at $u = 0, \infty$

$\Rightarrow A(u, u^{-1}) = cu^n$ for some $n \in \mathbb{Z}$.

Each such matrix gives rise to a line bundle, traditionally denoted $\mathcal{O}(-n)$. This is a complete classification.

What about general vector bundles? First, which $A(u, u^{-1})$ give rise to isomorphic vector bundles? The answer is the v.b. determined by the matrix $V(u^{-1}) A(u, u^{-1}) \circ U(u)$ is isomorphic to $A(u, u^{-1})$!

$n \times n$ matrices with constant determinant (scalar)

Can we reduce to canonical form? Every $A(u, u^{-1})$ is equivalent to a diagonal matrix with nonzero entries of the form u^{n_i}

(Grothendieck splitting Thm) **Cor.** Every vector bundle is a $\oplus$ of line bundles over $\mathbb{P}^1$. This holds over every field.

Coherent Sheaves

Let X be a noetherian scheme.

Defn. $\mathcal{F}$ a sheaf over X is called quasi-coherent if for any $U = \operatorname{Spec} A$ an open affine, $\mathcal{F}|_U = \widetilde{M}$ where M is an A-module.

We say it is coherent if M is finitely presented.

Example. $X = \operatorname{Spec} A$, $\mathcal{F} = \widetilde{M}$

$$\Gamma(X, -) : Q\,\mathrm{Coh}(X) \rightleftarrows \mathrm{Mod}(A) : \widetilde{(-)}$$

$$\mathrm{Coh}(X) \rightleftarrows \text{f.p. } \mathrm{Mod}(A)$$

Prop. $\mathrm{Coh}(X)$ forms an abelian category.

Sketch of Proof. • 0-object $\underline{0}(U) = 0$ ✓

· Direct sum $(\mathcal{F} \oplus \mathcal{G})(U) := \mathcal{F}(U) \oplus \mathcal{G}(U)$ ✓ This is a colimit! But modules don't know the difference between finite colimit and limit.

· $\ker(\mathcal{F} \to \mathcal{G})(U) = \ker(\mathcal{F}(U) \to \mathcal{G}(U))$ ✓

· $\operatorname{coker}(\mathcal{F} \to \mathcal{G})(U)$

$$\begin{tikzcd} & \mathcal{F}(U_1) \arrow[dr] & \\ \mathcal{F}(U_1 \cup U_2) \arrow[ur] \arrow[dr] & & \mathcal{F}(U_1 \cap U_2) \\ & \mathcal{F}(U_2) \arrow[ur] & \end{tikzcd}$$

should be pullback diagram

<u>Exercise.</u> If you are bored by this, try to prove category of homotopy sheaves over stable ∞-cat is again ∞-cat.

<u>Locally free sheaf.</u>

<u>Defn.</u> $\mathcal{F}$ is called <u>locally free</u> if there is an open affine cover $\{U_i = \operatorname{Spec} A_i\}$ such that

$$\mathcal{F}|_{U_i} = (\mathcal{O}_X|_{U_i})^{\oplus n} \quad \text{for some } n$$

It is called a <u>line bundle</u> if $n=1$.

<u>Note.</u> R is a ring of dim n, R is smooth

finite resolution

$$0 \to F_n \to \cdots \to F_1 \to F_0 \to M \to 0$$

This is Hilbert theorem.

Similarly, if $\mathcal{F}$ is a coherent sheaf on $\mathbb{P}^n$

$$0 \to \mathcal{E}_1 \to \cdots \to \mathcal{E}_0 \to F \to 0$$

and if $n=1$

$$0 \to \mathcal{E}_1 \to \mathcal{E}_0 \to \mathcal{F} \to 0$$

How do line bundles on $\mathbb{P}^n$ look like?

<u>$n=1$.</u> given $\mathbb{P}^1$, there's a diagram

$$\begin{tikzcd} k[x] \arrow[dr] & & k[y] \arrow[dl] \\ & k[\xi,\xi^{-1}] & \end{tikzcd}$$

Data of a line bundle on $\mathbb{P}^1$ is: a locally free rank 1 module on $k[x]$, a locally free rank 1 module on $k[y]$ and "clutching function" between them. There's actually great simplification,

<u>Thm</u> (Serre-Swan). For

locally free <=> projective.

In addition, projective modules over a PID are free. Can replace locally free by free.

<u>Rmk (Miro).</u>

We are using something very special for $n=1$.

So, $\mathcal{L}$ a line bundle. In sheaf language

free rank 1 $k[x]$-module call generator g_x $\to$ $\mathcal{L}(U_x)$

free rank 1 $k[y]$-module call generator g_y $\to$ $\mathcal{L}(U_y)$

$$\begin{tikzcd} \mathcal{L}(U_x) \arrow[dr] & & \mathcal{L}(U_y) \arrow[dl] \\ & \mathcal{L}(U_x \cap U_y) & \end{tikzcd}$$

$g_x \mapsto g_z$, $\quad g_y \mapsto A(x,x^{-1})g_z$

free $k[x,x^{-1}]$ module, call generator g_z.

some multiple of the generator $g_z = im(g_x)$.

$U_x = \operatorname{Spec} k[x]$

$U_y = \operatorname{Spec} k[y]$

We recall $A(x,x^{-1})$ can be assumed to be x^{-n}, $n \in \mathbb{Z}$.

So we've reproved

__Prop.__ $\forall n \in \mathbb{Z}$, $\exists!\ \mathcal{O}(n)$ up to iso. These are all the line bundles.

__Global sections.__ $\{(g_x, y^n g_y), (x g_x, y^{n-1} g_y), \text{——}, \text{——}, (x^n g_x, g_y)\}$ $n+1$ sections. So sections of $\mathcal{O}(n)$ correspond to deg n homogeneous elements of $k[x,y]$.

should make sense from chern classes or just by lookin[g] at trans[ition] functions.

S^1-action on G_m action

More generally, if S is a graded ring

Defn. $\mathcal{O}(1) = \widetilde{M}$ where $M_n = S_{n+1}$. Similarly

$\mathcal{O}(d) = \widetilde{M}$ where $M_n = S_{n+d}$. So

$\Gamma(X, \mathcal{O}(d)) =$ deg 0 elements of M $=$ deg d elements of S.

shifting degree down by d.

Inspiration for next discussion should be chern classes.

Prop. $\mathcal{O}(m) \otimes \mathcal{O}(n) \cong \mathcal{O}(m+n)$.

Proof. $\mathcal{O}(m) = \widetilde{S(m)}$

$\mathcal{O}(n) = \widetilde{S(n)}$

should make sense from chern classes or just by looking at transition functions.

Then $\widetilde{S(m)} \otimes \widetilde{S(n)} = \widetilde{S(m) \otimes S(n)} = \widetilde{S(m+n)} = \mathcal{O}(m+n)$. ∎

Defn. Given two sheaves F, G we have inner hom

$$\mathcal{H}om(F,G)(U) = \mathrm{Hom}(F|_U, G|_U)$$

In particular,

$$\Gamma(X, \mathcal{H}om(-,-)) = \mathrm{Hom}(-,-)$$

There's also an adjunction with tensor products.

Defn. The dual is $\mathcal{F}^\vee = \mathcal{H}om(\mathcal{F}, \mathcal{O}_X)$.

Prop. $\mathcal{O}(d)^\vee = \mathcal{O}(-d)$.

Proof. $\mathcal{O}(d) \otimes \mathcal{H}om(\mathcal{O}(d), \mathcal{O}_X) \longrightarrow \mathcal{O}_X$, check it is iso on stalk level and use adjunction. ⊠

Prop. $\mathcal{H}om(\mathcal{O}(m), \mathcal{O}(n)) \cong \mathcal{O}(n-m)$.

Proof. $\mathcal{H}om(\mathcal{O}(m), \mathcal{O}(n)) \cong \mathcal{H}om(\mathcal{O}(m), \mathcal{H}om(\mathcal{O}(-n), \mathcal{O}))$

$\cong \mathcal{H}om(\mathcal{O}(m) \otimes \mathcal{O}(-n), \mathcal{O}) = \mathcal{O}(n-m)$

Cor. $\mathrm{Hom}(\mathcal{O}(m), \mathcal{O}(n)) \cong \Gamma(X, \mathcal{O}(n-m))$

so there's some kind of filtration... you have morphisms but not the other way. Sounds suspiciously like Bridgeland stability condition.

Exercise. $\mathcal{O} \xrightarrow{f} \mathcal{O}(n)$ where $n>0$, f homogeneous of degree n. Show this is a monomorphism. Show the category of is finite length.

So we've constructed this line bundles, but not every coherent sheaf is locally free, of course.

Example. $f \in k[x,y]$ homogeneous of degree n as we've said

$$0 \to \mathcal{O} \xrightarrow{f} \mathcal{O}(n)$$

is a monomorphism. But what's the kernel?

Denote it $\mathcal{O}_f$. Then we have a SES

$$0 \to \mathcal{O} \xrightarrow{f} \mathcal{O}(n) \longrightarrow \mathcal{O}_f \to 0$$

Turns out f is a skyscraper sheaf!

For example take $f = x - y$, get skyscraper concentrated at $x = y$. More generally

torsion sheaf, not skyscraper neccesarily.

$\mathcal{O}_f$ is concentrated at the zeros of f.

Over every zero it is a v.f. of rank = multiplicity of f. Because $\mathcal{O}_f = \overbrace{k[x,y]}/(f)$

think of this torsion sheafs as "very small".

30~

<u>Thm.</u> $\mathcal{F}$ coherent sheaf on $\mathbb{P}^1$. Then $\mathcal{F}$ decomposes as direct sum of sheaves $\mathcal{O}(d)$ and $\mathcal{O}_f$.

<u>Exercise.</u> Show $\mathcal{O}_{(x-y)^2} \not\cong \mathcal{O}_{(x-y)} \oplus \mathcal{O}_{(x-y)}$

so this torsion ~~[illegible]~~ sheafs are <u>not</u> sum of skyscraper sheafs.

<u>(Derived functor) Cohomology</u>

There is a functor $\Gamma_{\mathbb{P}^1}$: coherent sheaf on $\mathbb{P}^1$ $\mathcal{F} \longmapsto \mathcal{F}(\mathbb{P}^1)$ global section

This is left exact, but unfortunatly not exact. So whenever we've got a SES of sheaves

$$0 \to A \to B \longrightarrow C \longrightarrow 0$$

we get

$$0 \to \Gamma_{\mathbb{P}^1} A \to \Gamma_{\mathbb{P}^1} B \to \Gamma_{\mathbb{P}^1} C \to$$

$$\hookrightarrow H^1(\mathbb{P}^1, A) \to H^1(\mathbb{P}^1, B) \to H^1(\mathbb{P}^1, C)$$

$$\hookrightarrow H^2(\mathbb{P}^1, A) \to \cdots$$

can extend!

How? Given a sheaf $\mathcal{F}$, resolve $\mathcal{F}$ by <u>injective sheaves</u> $0 \to \mathcal{F} \to \mathcal{I}^0 \to \mathcal{I}^1 \to \mathcal{I}^2 \to \cdots$

The functor it ~~re~~represents is exact.

get

$$0 \to \Gamma(\mathbb{P}^1, \mathcal{I}^0) \to \Gamma(\mathbb{P}^1, \mathcal{I}^1) \to \cdots$$

Define $H^i(\mathbb{P}^1, \mathcal{F})$ to be cohomology of this sequence.

Cech Cohomology

X noetherian and seperated (i.e. intersection of two affines is affine).

$\{U_i\} = \mathcal{U}$ is an affine open cover of X.

<u>Defn.</u> The <u>Cech resolution</u> (I don't know if this is standard terminology)

$$0 \to \mathcal{F} \to \prod_{i_0} \mathcal{F}|_{U_{i_0}} \to \prod_{i_0 < i_1} \mathcal{F}|_{U_{i_0} \cap U_{i_1}} \to \cdots$$

where the maps

$$\prod_{i_0 < i_1 < \cdots < i_p} \mathcal{F}|_{U_{i_0} \cap \cdots \cap U_{i_p}} \xrightarrow{d} \prod_{i_0 < \cdots < i_{p+1}} \mathcal{F}|_{U_{i_0} \cap \cdots \cap U_{i_{p+1}}}$$

$$(\partial s)_{i_0, i_1, \dots, i_{p+1}} = \sum_{k=0}^{p+1} (-1)^k (s_{i_0, i_1, \dots, i_{p+1}})\Big|_{U_{i_0} \cap \dots \cap U_{i_{p+1}}}$$

$$\check{H}^i(X, \mathcal{F}) = H^i(\Gamma(X, \text{the complex above}))$$

<u>Model structure explanation.</u>