

280x (15/10)

let's try to understand v.b. on \mathbb{P}^1 :

\mathbb{P}^1 has open cover by $A^1 = \text{Spec } k[s]$

$$A^1 = \text{Spec } k[t]$$

and gluing

$$A^1 - \{0\} = \text{Spec } k[t, t^{-1}] \rightarrow \text{Spec } k[s, s^{-1}] = A^1 - \{\infty\}$$

$s \mapsto t^{-1}$

A vector bundle is determined by gluing

data. That is,

$$(u, v) \mapsto (u^{-1}, A(u, u^{-1})v)$$

\uparrow
 $n \times n$ matrix with entries Laurent polynomial

Condition on A: $\det A(u, u^{-1}) \neq 0$ for all $u \neq 0, \infty$.

Restrict to line bundles. Then $A(u, u^{-1}) = 1 \times 1$

matrix i.e. Laurent polynomial it does not

vanish except possibly at $u=0, \infty$

$\Rightarrow A(u, u^{-1}) = u^n$ for some $n \in \mathbb{Z}$.

Each such matrix gives rise to a line bundle, traditionally denoted $\mathcal{O}(-n)$. This is a complete classification.

What about general vector bundles? First, which $A(u, u^{-2})$ give rise to isomorphic vector bundles? The answer is the v.b. determined by the matrix $V(u^{-2})A(u, u^{-2})U(u)$ is isomorphic to $A(u, u^{-2})$!

\nwarrow \nearrow
 $n \times n$ matrices
with constant
determinant
(scalar)

Can we reduce to canonical form? Every $A(u, u^{-2})$ is equivalent to a diagonal matrix with nonzero entries of the form u^{n_i} .

(another
splitting
thm)

Cor. Every vector bundle is a \oplus of line bundles over \mathbb{P}^1 . This holds over every field.

Coherent Sheaves

Let X be a noetherian scheme.

Defn. \mathcal{F} a sheaf over X is called quasi-coherent

if for any $U = \text{Spec } A$ an open affine,

$$\mathcal{F}|_U = \tilde{M} \text{ where } M \text{ is an } A\text{-module.}$$

We say it is coherent if M is finitely presented.

Example. $X = \text{Spec } A, \mathcal{F} = \tilde{M}$

$$\Gamma(X, -) : \mathcal{C}oh(X) \xleftrightarrow{\sim} \text{Mod}(A) : (-)$$

$$\mathcal{C}oh(X) \xleftrightarrow{\sim} \text{f.p. Mod}(A)$$

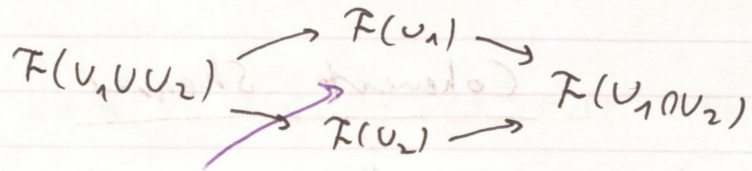
Prop. $\mathcal{C}oh(X)$ forms an abelian category.

Sketch of proof. • 0-object $\mathcal{O}(U) = 0 \vee$

• Direct sum $(\mathcal{F} \oplus \mathcal{G})(U) := \mathcal{F}(U) \oplus \mathcal{G}(U) \checkmark$ *This is a colimit! But modules don't know the difference between finite colimit and limit.*

• $\ker(\mathcal{F} \rightarrow \mathcal{G})(U) = \ker(\mathcal{F}(U) \rightarrow \mathcal{G}(U)) \checkmark$

• $\text{coker}(\mathcal{F} \rightarrow \mathcal{G})(U)$



should be pullback diagram

Exercise. If you are bored by this, try to prove category of homotopy sheaves over stable ∞ -cat is again ∞ -cat. Locally free sheaf.

Defn. \mathcal{F} is called locally free if there is an

open affine cover $\{U_i = \text{Spec } A_i\}$ such that

$$\mathcal{F}|_{U_i} = (\mathcal{O}_X|_{U_i})^{\oplus n} \text{ for some } n$$

It is called a line bundle if $n=1$.

Note. R is a ring of ∞ -lim n , R is smooth

finite resolution $\xrightarrow{\quad}$

$$0 \rightarrow F_n \rightarrow \dots \rightarrow F_1 \rightarrow F_0 \rightarrow M \rightarrow 0$$

This is Milbert theorem.

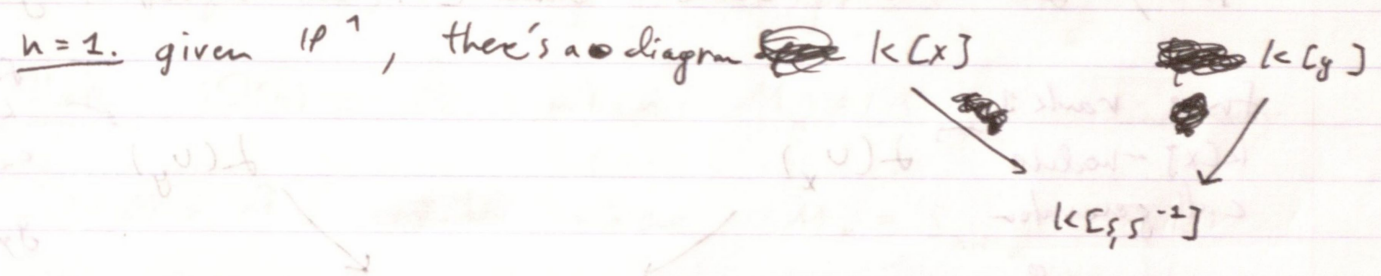
Similarly, if \mathcal{F} is a coherent sheaf on \mathbb{P}^n

$$0 \rightarrow \mathcal{E}_1 \rightarrow \dots \rightarrow \mathcal{E}_n \rightarrow \mathcal{F} \rightarrow 0$$

and if $n=1$

$$0 \rightarrow \mathcal{E}_1 \rightarrow \mathcal{E}_0 \rightarrow \mathcal{F} \rightarrow 0$$

How do line bundles on P^n look like?



Data of a line bundle on P^1 is a locally free rank 1 module on $k[x]$, a locally free rank 1 module on $k[y]$ and "Clutching function" between them. There's actually great simplification,

Thm (Serre-Swan). For

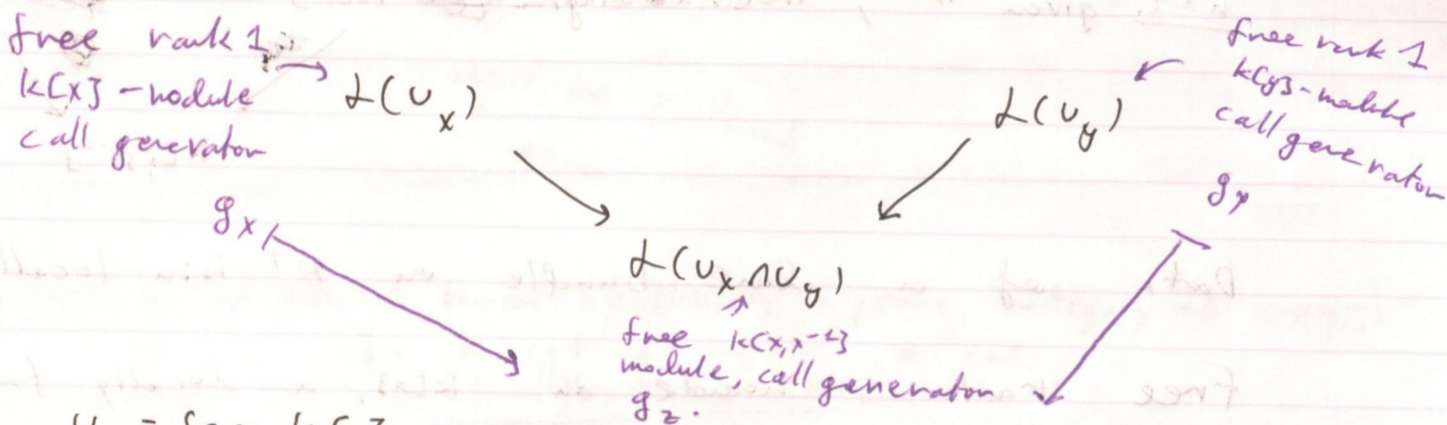
locally free \Leftrightarrow projective.

In addition, projective modules over a PID are free. (can replace locally free by free.)

Rank (Miro).

We are using something very special for $n=1$.

$\mathcal{O}(1)$ a line bundle. In sheaf language



$U_x = \text{Spec } k[x]$

$U_y = \text{Spec } k[y]$

$A(x, x^{-1})g_z$
some multiple of the generator $g_z = \text{img}(g_x)$.

We recall $A(x, x^{-1})$ can be assumed to be $x^{-n}, n \in \mathbb{Z}$.

So we've reproved

Prop. $\forall n \in \mathbb{Z}, \exists! \mathcal{O}(n)$ up to iso. These are all the line bundles.

Global sections. $\{ (g_x, y^n g_y), (x g_x, y^{n-1} g_y), \dots, (x^n g_x, g_y) \}$ $n+1$ sections. So sections of $\mathcal{O}(n)$ correspond to deg n homogeneous elements of $k[x,y]$.

should make sense from chain classes or just by looking at transition functions.

More generally, if S is a graded ring S^1 -action on G_m action

Defn. $\mathcal{O}(1) = \tilde{M}$ where $M_n = S_{n+1}$. Similarly

$\mathcal{O}(0) = \tilde{M}$ where $M_n = S_{n+d}$. So

$\Gamma(X, \mathcal{O}(d)) =$ deg 0 elements of M = deg d elements of S .
 \uparrow shifting degree down by d .

In spiration for next discussion should be Chern classes.

Prop. $\mathcal{O}(m) \otimes \mathcal{O}(n) \cong \mathcal{O}(m+n)$.

Proof. $\mathcal{O}(m) \cong \tilde{S}(m)$

$\mathcal{O}(n) \cong \tilde{S}(n)$

Then $\tilde{S}(m) \otimes \tilde{S}(n) = \tilde{S}(m) \otimes \tilde{S}(n) = \tilde{S}(m+n) = \mathcal{O}(m+n)$.

Defn. Given two sheaves F, G we have inner hom

$\text{Hom}(F, G)(U) = \text{Hom}(F|_U, G|_U)$

In particular,

$\Gamma(X, \text{Hom}(-, -)) = \text{Hom}(-, -)$

There's also an adjunction with tensor products.

should make sense from Chern classes or just by looking at transition functions.

Defn. The dual is $\mathcal{F}^\vee = \text{hom}(\mathcal{F}, \mathcal{O}_X)$.

Prop. $\mathcal{O}(d)^\vee = \mathcal{O}(-d)$.

Proof. $\mathcal{O}(d) \otimes \text{hom}(\mathcal{O}(d), \mathcal{O}_X) \rightarrow \mathcal{O}_X$, check it is iso on stalk level and use adjunction. \square

Prop. $\text{hom}(\mathcal{O}(m), \mathcal{O}(n)) \cong \mathcal{O}(n-m)$.

Proof. $\text{hom}(\mathcal{O}(m), \mathcal{O}(n)) \cong \text{hom}(\mathcal{O}(m), \text{hom}(\mathcal{O}(-n), \mathcal{O}))$
 $\cong \text{hom}(\mathcal{O}(m) \otimes \mathcal{O}(-n), \mathcal{O}) = \mathcal{O}(n-m)$

Con. $\text{Hom}(\mathcal{O}(m), \mathcal{O}(n)) \cong \Gamma(X, \mathcal{O}(n-m))$

so there's some kind of filtration... you have

monomorphisms

but not the

other way. Sounds suspiciously like Bridgeland

stability condition.

Exercise. $\mathcal{O} \xrightarrow{f} \mathcal{O}(n)$ where $n > 0$, f homogeneous

of degree n . Show this is a monomorphism. Show

the category of $\mathcal{O} \xrightarrow{f} \mathcal{O}(n)$ is

finite length.

torsion
sheaf,
not
slyso
necco

So we've constructed this line bundle, but not every coherent sheaf is locally free, of course.

Example. $f \in k[x, y]$ homogeneous of degree n as we've said

$$0 \rightarrow \mathcal{O} \xrightarrow{f} \mathcal{O}(n)$$

is a monomorphism. But what's the kernel?

Denote it \mathcal{O}_f . Then we have a SES

$$0 \rightarrow \mathcal{O}_f \xrightarrow{f} \mathcal{O}(n) \rightarrow \mathcal{O}_f \rightarrow 0$$

Turns out \mathcal{O}_f is a skyscraper sheaf.

For example take $f = x - y$, get skyscraper concentrated at $x = y$. More generally

\mathcal{O}_f is concentrated at the zeros of f .

torsion sheaf, not skyscraper necessarily.

Over every zero it is a v.f. of rank = multiplicity of f . Because $\mathcal{O}_f = \overbrace{k[x, y]} / (f)$

think of this torsion sheaf as "very small".

Thm. \mathcal{F} coherent sheaf on \mathbb{P}^1 . Then \mathcal{F} decomposes as direct sum of sheaves $\mathcal{O}(d)$ and \mathcal{O}_f .

Exercise. Show $\mathcal{O}_{(x-y)^2} \not\cong \mathcal{O}_{(x-y)} \oplus \mathcal{O}_{(x-y)}$

so this torsion sheaf are not sum of skyscraper sheafs.

(Derived Functor) Cohomology

There is a functor $\Gamma_{\mathbb{P}^1}$: coherent sheaf on $\mathbb{P}^1 \rightarrow \mathbb{R}(\mathbb{P}^1)$ section

This is left exact, but unfortunately not exact. So whenever we've got a SES of sheaves

$$0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$$

we get

$$0 \rightarrow \Gamma_{\mathbb{P}^1} A \rightarrow \Gamma_{\mathbb{P}^1} B \rightarrow \Gamma_{\mathbb{P}^1} C \rightarrow \dots$$

$$\rightarrow H^1(\mathbb{P}^1, A) \rightarrow H^1(\mathbb{P}^1, B) \rightarrow H^1(\mathbb{P}^1, C) \rightarrow \dots$$

$$\rightarrow H^2(\mathbb{P}^1, A) \rightarrow \dots$$

can extend!

How? Given a sheaf \mathcal{F} , resolve \mathcal{F} by injective

sheaves $0 \rightarrow \mathcal{F} \rightarrow \mathcal{I}^0 \rightarrow \mathcal{I}^1 \rightarrow \mathcal{I}^2 \rightarrow \dots$

The functor
it represents
is exact.

get injective sheaves

$0 \rightarrow \Gamma(P^1, \mathcal{I}^0) \rightarrow \Gamma(P^1, \mathcal{I}^1) \rightarrow \dots$

Define $H^i(P^1, \mathcal{F})$ to be cohomology of this sequence.

Cech Cohomology

X noetherian and separated (i.e. intersection of two affines is affine).

$\{U_i\} = \mathcal{U}$ is an affine open cover of X .

Defn. The Cech resolution (I don't know if this is standard terminology)

$0 \rightarrow \mathcal{F} \rightarrow \prod_{i_0} \mathcal{F}|_{U_{i_0}} \rightarrow \prod_{i_0 < i_1} \mathcal{F}|_{U_{i_0} \cap U_{i_1}} \rightarrow \dots$

where the maps

$\prod_{i_0 < i_1 < \dots < i_p} \mathcal{F}|_{U_{i_0} \cap \dots \cap U_{i_p}} \xrightarrow{d} \prod_{i_0 < \dots < i_{p+1}} \mathcal{F}|_{U_{i_0} \cap \dots \cap U_{i_{p+1}}}$

$$(d_s)_{i_0, i_1, \dots, i_{p+1}} = \sum_{k=0}^{p+1} (-1)^k (s_{i_0, i_1, \dots, i_{p+1}}) \Big|_{v_{i_0} \dots v_{i_{p+1}}}$$

$$\check{H}^i(X, \mathbb{F}) = H^i(\text{the complex above})$$

Model structure explanation.