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280x (17/10)

Motivation One of the important statement we'll see is Serre duality: X smooth proj/k of dim n. $\mathcal{F}$ locally free then $H^i(X, \mathcal{F}^\vee \otimes \omega_X) \cong H^{n-i}(X, \mathcal{F})^*$

$\uparrow$ sheaf of differential forms

That's what we'll see explicitly, by analyzing vector bundles on $\mathbb{P}^n$.

last time, for X noetherian and seperated we've defined for any open cover $\mathcal{U} = \{U_i\}$

$$0 \to \mathcal{F} \to \prod_{i_0} \mathcal{F}|_{U_{i_0}} \to \prod_{i_0 < i_1} \mathcal{F}|_{U_{i_0} \cap U_{i_1}} \to \cdots$$

Functor is $\Gamma(X, -)$. So every sheaf $\mathcal{F}|_{U_{i_0} \cap \cdots \cap U_{i_p}}$ is supported on open affine.

distinguished open affine $U_i = D(x_i)$

Take $X = \mathbb{P}^n_k$, let $\mathcal{U} = (U_0, U_1, \ldots, U_n)$

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Thm. 1) $H^i(X, \mathcal{O}(d)) = 0$ if $i \neq 0, n$

2) $H^n(X, \mathcal{O}(-n-1)) \cong k$ turn out $\omega_X = \mathcal{O}(-n-1)$

3) There's a natural pairing between

$$H^0(X, \mathcal{O}(d)) \times \underbrace{H^n(X, \mathcal{O}(-d-n-1))} \to H^n(X, \mathcal{O}(-n-1)) \cong k$$

is a perfect pairing.

$H^n(X, \mathcal{O}(d)^\vee \otimes \mathcal{O}(-n-1))$

<u>Derived Functors</u>

$F: A \to B$ left exact. Can define R^iF the i^{th} right derived functor: Resolve X by injectives

$$0 \to X \to I^0 \to I^1 \to I^2 \to \cdots$$

Apply F and drop X

$$0 \to F(I^0) \to F(I^1) \to F(I^2) \to \cdots$$

That's not an exact sequence in general. Define the homology at I^i to be $R^iF(X)$.

Examples. (1) The derived functors of

$$\Gamma : \left\{ \begin{array}{c} \text{coherent} \\ \text{sheaves on } \mathbb{P}^k \end{array} \right\} \longrightarrow Ab$$

are cohomology of sheaves.

(2) If $\mathcal{F}$ is a coherent sheaf

$$\mathcal{H}om_{\text{sheaf}}(\mathcal{F}, -)$$

are known as $\mathcal{E}xt^i(\mathcal{F}, -)$

Hom sheaf

(3) $\Gamma(\mathcal{H}om_{\text{sheaf}}(\mathcal{F}, -)) = \mathrm{Hom}(\mathcal{F}, -)$

Derived functors are $\mathrm{Ext}^i(\mathcal{F}, -)$

Caution: Different!

$\mathcal{E}xt^i(\mathcal{O}(n), \mathcal{O}(m))$ & $\mathrm{Ext}^i(\mathcal{O}(n), \mathcal{O}(m))$

$\mathcal{E}xt^i(\mathcal{O}(n), \mathcal{O}(m)) = 0$ for $i > 0$ because $\mathcal{O}(n)$ proj.

There's a spectral sequence

$$E_2^{i,j} = H^i(X, \mathcal{E}xt^j(\mathcal{O}(n), \mathcal{O}(m))) \Rightarrow$$

$$\mathrm{Ext}^{i+j}(\mathcal{O}(n), \mathcal{O}(m))$$

So $\mathrm{Ext}^i(\mathcal{O}(n), \mathcal{O}(m)) = H^i(\mathcal{O}(n-m), \mathbb{P}^k)$.

locally free

Upshot. $\mathrm{Ext}^i(\mathcal{F}, \mathcal{L} \otimes \mathcal{G}) \cong \mathrm{Ext}^i(\mathcal{F} \otimes \mathcal{L}^\vee, \mathcal{G})$

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Why is $\omega_X \cong \mathcal{O}(-n-1)$?

$\Omega_{X/k}$ is called the sheaf of differentials.

Locally given by symbols dx.

Example. On $X = \mathbb{P}^n$ get coordinate on U_i:

$$k[x_{0i}, x_{1i}, \dots, \hat{x}_{ii}, \dots, x_{ni}]$$

think x_0/x_i

Then $\Omega_{X/k}$ is the free $k[x_{0i}, x_{1i}, \dots, \hat{x}_{ii}, \dots, x_{ni}]$-module over $dx_{0i}, dx_{1i}, \dots, \widehat{dx}_{ii}, \dots, dx_{ni}$.

Def. Let X of dim n. We define

$$\omega_X = \bigwedge^n \Omega_{X/k}$$

That's sheaf of differential n-forms.

Example. In $X = \mathbb{P}^n$ it is locally free rank 1 module over $dx_{0i} \wedge \dots \widehat{dx}_{ii} \wedge \dots \wedge dx_{ni}$

<u>Example.</u> Calculate transition function from U_0 to U_1:

$$d(x_{10}) \wedge d(x_{20}) \wedge \ldots \wedge d(x_{n0})$$

$$= d\left(\frac{1}{x_{01}}\right) \wedge d\left(\frac{x_{21}}{x_{01}}\right) \wedge \ldots \wedge d\left(\frac{x_{n1}}{x_{01}}\right)$$

$$= -\frac{x_{01}}{x_{01}^2} \wedge \bigwedge_{i=2}^{n} \left(\frac{dx_{0i}}{x_{01}}\right) = -\frac{1}{x_{01}^{n+1}} dx_{01} \wedge dx_{21} \wedge \ldots \wedge dx_{n1}$$

$$\Rightarrow \omega_X \cong \mathcal{O}(-n-1)$$

<u>Serre duality.</u> $X = \mathbb{P}^n_k$, $\mathcal{F}$ coherent.

1) $H^n(X, \omega_X) \cong k$ ✓ computation

2) $\mathrm{Hom}_{\mathcal{O}}(\mathcal{F}, \omega_X) \times H^n(X, \mathcal{F}) \to H^n(X, \omega_X) \cong k$

is a perfect pairing

3) $\mathrm{Ext}^i_{\mathcal{O}_X}(\mathcal{F}, \omega_X) \xrightarrow{\sim} H^{n-i}(X, \mathcal{F})^*$

<u>Break down of proof.</u> a) Do it for line bundles

(that's basically the computation we've done

$$\mathrm{Ext}^i(\mathcal{F}, \omega_X) \cong \mathrm{Ext}^i(\mathcal{O}_X, \mathcal{F}^\vee \otimes \omega_X)$$

$$= H^i(X, \mathcal{F}^\vee \otimes \omega_X)$$

So we should believe it for line bundles. b) Next, if we have

$$0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$$

and the statement holds for two of the terms, it holds for the third.

Why is that? Well, we get LES

$$\begin{tikzcd} 0 \arrow[r] & H^0(X,\mathcal{F}) \arrow[r] & H^0(X,\mathcal{G}) \arrow[r] & H^0(X,\mathcal{H}) \arrow[dll, out=-10, in=170] \\ & H^1(X,\mathcal{F}) \arrow[r] & H^1(X,\mathcal{G}) \arrow[r] & H^1(X,\mathcal{H}) \arrow[dll, out=-10, in=170] \\ & \cdots & & \end{tikzcd}$$

we get another LES, use properties of δ-functor and 5-lemma.

c) If you have

$$0 \to \mathcal{E}_n \to \mathcal{E}_{n-1} \to \cdots \to \mathcal{E}_0 \to \mathcal{F} \to 0$$

can decompose into SES because

$$
\begin{tikzcd}
 & & \operatorname{coker} d_n \arrow[dr] & & & \\
0 \arrow[r] & \mathcal{E}_n \arrow[r, "d_n"] & \mathcal{E}_{n-1} \arrow[u] \arrow[r] & \mathcal{E}_{n-2} \arrow[rr] \arrow[dr] & & \mathcal{E}_{n-1} \arrow[r] & \cdots \\
 & & & & \operatorname{coker} d_{n-1} \arrow[ur] & &
\end{tikzcd}
$$

SES

SES

d) $\mathcal{F}$ can be resolved by locally free sheaf.

?. e) Any locally free sheaf has a [redacted] sub line-bundle.

<u>Derived World</u>

Have an iso $\operatorname{Ext}^i(\mathcal{F}, \omega_X) \xrightarrow{\sim} H^{n-i}(X, \mathcal{F})^*$

$$\overset{?}{\cong} \quad H^i(X, \mathcal{F}^\vee \otimes \omega_X)$$

If you consider $S : D^b \operatorname{Coh} X \longrightarrow D^b \operatorname{Coh} X$ additive

$$\mathcal{F} \longmapsto \mathcal{F} \otimes \omega_X [n]$$

that's a Serre duality functor:

$$
\begin{tikzcd}
\operatorname{Hom}_D(A, B) \arrow[r, "\sim"] \arrow[rr, bend right, "S"'] & \operatorname{Hom}_D(B, SA) \arrow[r] & \operatorname{Hom}_D(SA, SB)
\end{tikzcd}
$$

because you can resolve every sheaf by locally free etc.

Tilting Sheaves

Recall the exercise from earlier the term

$$X = \mathbb{P}^1_k, \qquad T = \mathcal{O} \oplus \mathcal{O}(1)$$

claimed:
- $\mathrm{Ext}^i(T,T) = 0$, $i \neq 0$
- $A = \mathrm{End}_{\mathcal{O}_X}(T) \simeq \mathrm{path}(\cdot \rightrightarrows \cdot)$
- $D^b\,\mathrm{Coh}\,\mathbb{P}^1 \xrightarrow{\sim} D^b\,\mathrm{Coh}(\mathrm{mod}\,A^{op}) = D^b\,\mathrm{Rep}(\cdot \rightrightarrows \cdot)$

this allows us to understand the derived category of sheaves as modules over algebra!

$$E \longmapsto \mathrm{Ext}^\bullet(\mathcal{O} \oplus \mathcal{O}(1), E)$$

Idea. Sheaves on X $\xrightarrow[\text{using } T]{}$ modules over A

Def. k alg closed (as usual), X smooth proj over k. $D = D^b\,\mathrm{Coh}\,X$. We say that $T \in \mathrm{Coh}\,X$ is a tilting sheaf if:

i.e. every module over A has finite projective resolution of length $\leq n$.

(T1) $A = \mathrm{End}_{\mathcal{O}_X}(T)$ has finite global dimension $= n$

(T2) $\mathrm{Ext}^i_{\mathcal{O}_X}(T,T) = 0$ for $i \neq 0$

(T3) T generates the category D under shifts, cones, direct summand, isos.

actually comes for free like direct sum...

Example. Get a SES of $X = \mathbb{P}^1$ sheaves:

$$0 \to \mathcal{O}(-1) \to \mathcal{O}^2 \to \mathcal{O}(1) \to 0$$

tensor with $\mathcal{O}(d)$ for $d \in \mathbb{Z}$

$$0 \to \mathcal{O}(d-1) \to \mathcal{O}(d)^2 \to \mathcal{O}(1+d) \to 0$$

get all $\mathcal{O}(n)$, $n \in \mathbb{Z}$, then we can get all sky scrapers...

upshot $T = \mathcal{O} \oplus \mathcal{O}(1)$ satisfy (T3).

Thm. Let T be a tilting sheaf. Have

$$F(-) := \mathrm{Hom}(T, -) : \mathrm{Coh}\, X \longrightarrow \text{f.p. mod}(A^{op})$$

right A-mod = left A^{op}-mod

$$G(-) := - \otimes_A T : \text{f.p. mod}(A^{op}) \longrightarrow \mathrm{Coh}\, X$$

$A = \mathrm{End}_{\mathcal{O}_X}(T)$

which give inverse equivalences

$$RF : D^b \mathrm{Coh}\, X \underset{\sim}{\overset{\sim}{\rightleftarrows}} D^b \mathrm{mod}(A^{op}) : LG$$

This is what give tilting sheaves their power.

Sketch. • Check that we land in D

For RF, need smoothness $\Rightarrow$ only finitely many $\mathrm{Ext}^i \neq 0$.

For LG, (T1) means A has finite global dimension.

· Using (T2) we see that

$$(RF \circ LG)(A) = RF(T) = A$$

all higher Ext vanish.

· Use (T3), image of LG, which is what's generated by T, is all of $D^b \operatorname{Coh} X$.

$\blacksquare$

<u>Thm.</u> $\mathcal{O} \oplus \mathcal{O}(1) \oplus \cdots \oplus \mathcal{O}(n) = T$ is a tilting sheaf on $\mathbb{P}^n$.

Can probably deduce that from

$$0 \to \mathcal{O}^{\binom{n+1}{0}} \to \mathcal{O}(1)^{\binom{n+1}{1}} \to \cdots \to \mathcal{O}(n)^{\binom{n+1}{n}} \to \mathcal{O}(n+1)^{\binom{n+1}{n+1}} \to 0$$

then can get

$$0 \to \mathcal{O}(1)^{\binom{n+1}{0}} \to \cdots \to \mathcal{O}(n+2)^{\binom{n+1}{n+1}} \to 0$$

same for tensoring with $\mathcal{O}(-1)$ to get negative line bundles. The Quiver with relations

$$\mathcal{O} \xrightarrow{x_0} \mathcal{O}(1) \xrightarrow{x_0} \cdots \xrightarrow{x_0} \mathcal{O}(n)$$